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The Review of Economic Studies, Vol. 58, No. 2. (Apr., 1991), pp. 259-276.

Review of Economic Studies (1991) 58, 259-276
0034-6527/91/00170259$02.00

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First version received May 1988; *final version accepted August* 1990 (*Eds.*)

This paper concerns the pattern of contributions to a joint project when commitments and enforceable contracts are not available. We analyse a game in which partners alternate in making contributions to the project until the project is completed. Contributions are sunk when they are made. The game has a unique subgame perfect equilibrium path, which is inefficient in the sense that socially desirable projects may not be completed. By contrast, in a "subscription game" in which the cost of the contribution is borne only if and when the contributions committed to the project cover its cost, the outcome is efficient.

1. INTRODUCTION

This paper addresses the following situation. A joint project is "socially desirable" in the sense that its total cost is lower than the total benefit that the partners will receive upon its completion. The project requires the contribution of resources by the partners which, once made, are sunk. Since the project is a public good, each partner would obviously like to free-ride on the contributions of others. Suppose it is impossible for partners to commit in advance to a specific sequence of contributions. That is, no enforceable contract can be written. The question is whether the project will be completed, and if so, what will be the pattern of the contributions by each partner.

To illustrate the problem, suppose it is agreed that one partner, call him A, will complete the first half of the project, and another partner, called B, will complete the project subsequently. After A has completed his part, making a sunk-cost investment, B can attempt to renegotiate over the remaining task pointing out that the sunk cost, being sunk, is irrelevant. Knowing this, A may be unwilling to complete his part before B makes some contribution as well.

The notion that the lack of an enforcement and commitment technology may present problems in various transactions is common to many problems in economics and other areas. Consider, for example the following discussion by Schelling (1960), which also suggests a possible approach to the problem.[1]

"If each party agrees to send a million dollars to the Red Cross on condition the other does, each may be tempted to cheat if the other contributes first, and each

1. We are grateful to Jeff Borland for bringing this discussion to our attention.

one's anticipation of the other's cheating will inhibit agreement. But if the contribution is divided into consecutive small contributions, each can try the other's good faith for a small price. Furthermore, since each can keep the other on short tether to the finish, no one ever need risk more than one small contribution at a time." (Schelling (1960, p. 45).)

The above discussion suggests that when enforcement is lacking, it is natural to think of a procedure in which agents alternate in making small steps towards the completion of the transaction, i.e. dividing the total "action" to smaller parts. In this paper we analyse such a procedure for the joint project problem. Specifically, we study a game in which at each stage one of the partners decides on the size of his contribution at this stage, until the project is completed.

The idea of overcoming the lack of commitment and enforcement by a procedure in which agents alternate in taking small steps is relevant in many other related applications. For example, consider an employer who hires a worker to perform some task. As in the case of a joint project, if the worker and the employer agree that the worker will first perform the task and the employer will pay subsequently then, absent enforcement mechanisms, the employer may not pay. Similarly, if payment is made in advance, the worker may not have incentives to perform the task. Again, what may alleviate the problem, and what seems to occur in reality, is that the employer and the worker take turns paying and working in small increments.[2]

Enforcement is particularly problematic and typically impossible when the parties involved are different nations. Not surprisingly, many international agreements take the form of alternating small steps. For example, the execution of the Camp David peace treaty between Israel and Egypt was designed to take the form of sequential alternating steps.

We can also find related examples in the context of biology and evolution. Smith (1981) describes the phenomenon of "egg trading" in the black hamlet, a coral reef fish.

"... These fish are simultaneous hermaphrodites with external fertilization. Mating occurs in pairs, each fish fertilizing the eggs laid by its partner.

... a pair of fish will come together and engage in a "spawning bout." During a typical bout with the same partner, the fish will engage in four to twelve spawning acts, about equally frequently as a male and as a female. Male and female roles alternate rather regularly during a bout. It is this alternate laying of eggs which is described as egg-trading.

To understand what is happening we must ask what behaviour is to be expected from a simultaneous hermaphrodite. If, as is likely, eggs are more expensive in resources than sperm, a hermaphrodite can increase its fitness by fertilizing the eggs of several other individuals, while insuring that its own eggs are fertilized. Thus, suppose that in a single spawning bout first one partner laid all its eggs and then the other. It would then pay each of these fish to pair with a fresh partner and to fertilize its eggs also. The new partner will be the loser, because it would have no eggs to fertilize. Also, since hamlets do not have mature eggs to lay every day, but could easily produce sperm every day, it would pay fish to pair on days when they

2. A similar procedure, in the context of a simple exchange between a buyer and a seller, is discussed in Hart and Holmstrom (1985), Section 3.4.

had no eggs ... egg trading evolved to prevent this kind of cheating." (Smith (1981), pp. 159–160.)[3]

Although this paper is tailored to the joint project example, our analysis sheds light on the related issues described above. A feature of the joint project problem which is important in this regard is that agents obtain a positive benefit from the actual completion of the project. In fact, in the model analysed here benefits are only obtained upon the completion of the project.[4]

As already mentioned, we analyse a contribution game in which partners take turns making contributions to the joint project. The cost of each contribution is borne by the relevant agent at the time the contribution is made. The benefit is obtained if and when the project is completed. At each stage the partner whose turn it is to contribute decides how much he would like to contribute in light of previous contributions and the amount still needed for completion. The model assumes complete and perfect information, and we seek the subgame perfect equilibria of the game.

We show that if players are impatient and their disutility from the contributions is a convex function, then the contribution game has essentially a unique subgame perfect equilibrium path. If the cost of the project is not too high, then the project is completed in equilibrium. Otherwise, the project is not completed, and in the unique equilibrium each player contributes nothing. We then show that the contribution game does not lead to an efficient outcome in the sense that socially desirable projects may not be completed. In fact, if the cost function for contributions is linear then the project is completed if and only if it would be completed by *one* agent if he is the only one. We also characterize the set of projects completed in equilibrium for the limit case in which the time between moves vanishes. We show that, at least for the set of power cost functions, many socially efficient projects are not completed in the equilibrium of the game.

The game which is the focus of our analysis captures a situation in which contributions are sunk at the time they are made, since commitments to make contributions later cannot be made. In order to understand how the lack of a commitment technology affects the results and particularly the inefficiency result, we analyse another procedure in which agents bear the cost of the contribution only when and if enough contributions are pledged to complete the project. In contrast to the result in the contribution game, we show that the unique subgame perfect equilibrium of this "subscription game" leads to an efficient outcome, in which all socially desirable projects are completed. In fact, the division of the cost in this model is analogous to that obtained in the bargaining game of Rubinstein (1982).

There is a large literature which is concerned with the provision of a public good and the associated free-rider problem. Within this very general context, the model we analyse is characterized by a number of structural assumptions. First, the public good (i.e. the project) is either provided or not, as compared with the more standard formulation where the question is the scale of the public good that is provided. Second, contributions are divisible, both among individuals and through time. And third, contributions are made in sequence, first by one player, then by the second, then by the first again, and so on.

3. For an interesting model that tackles a somewhat different aspect of the behaviour of the black hamlet, and for a discussion and references to the field observations of the fish, see Friedman and Hammerstein (1989). Interestingly, another fish, the harequin bass, which forms long-term mating relationships, uses only a two-round spawning procedure. This illustrates that long-term relationships can serve as an enforcement device. (We are grateful to Jim Friedman for bringing the bass example, as well as his paper with Hammerstein, to our attention.)

4. An interesting question, which we do not address here, is how the procedure is chosen when commitments cannot be made. Our analysis takes the procedure itself, i.e. the rules of the game, as given.

A number of papers investigate similar general questions in contexts that change our structural assumptions in at least one respect. Among them are the following. Fershtman (1986) considers a dynamic game in which players contribute over time to a stock of public capital which is subject to depreciation. Using the methodology of differential games, Fershtman presents an equilibrium in which there is underinvestment (reminiscent of our results), although he does not obtain uniqueness of his equilibrium. Bagnoli and Lipman (1989) discuss the provision of a public good by a one-shot simultaneous-move "contribution" game with a dichotomous decision whether the public good is provided or not. In this game contributions are returned to individuals if their total is not sufficiently high. In a sense, this n-player Nash demand game is a one-shot variation of the "subscription" game we discuss in Section 5. They obtain an efficiency result which complements ours. In Bliss and Nalebuff (1984), the game is one of waiting to see who will bear the entire (indivisible) cost of providing the public good. Finally, Varian (1989) considers the standard public good problem where the issue is the scale of provision of the good, contrasting situations where contributions are made simultaneously (and at one time only) with a situation where one participant can commit to his contribution first. By assumption, there is only one round of contributions and no discounting takes place.

Our model and especially the solution techniques are closest to the work of Harris and Vickers (1985), who analyse a model of a patent race. In the language of our model one can think of the patent race model as a situation in which the benefit from the project only goes to the first player whose total investments exceed a given level. Harris and Vickers show that in the unique equilibrium of the patent race game the behaviour of the winner is almost identical to what it would be if he is the only player. The other player invests nothing. In the equilibrium of our contribution game, by contrast, players alternate in making positive contributions to the project.

The paper is organized as follows. In Section 2 we describe the contribution game. Section 3 provides the analysis of the subgame perfect equilibria of the game and includes the main result of the paper. In Section 4 we examine the efficiency properties of the equilibrium. In Section 6 we analyse an alternative "subscription" game. The characterization of the set of completed projects for the limit case in which the time between moves vanishes is included in Appendix A.

2. THE CONTRIBUTION GAME

We focus on the following simple game with complete and perfect information. There are two identical players, player 1 and 2.[5] Each player values the immediate completion of a project by V. The cost of completing the project is K. Whether (and when) the project is completed depends on the contributions made by the players over time, as discussed below.

Players take turns in making contributions, starting with player 1 in period 1. The project is completed as soon as the total contributions made by both players reaches the total cost K.

Let c_i^t be the amount of player i's contribution at period t. A *history* at time τ is a sequence of contributions $(\{c_1^t\}_{t=1}^{\tau-1}, \{c_2^t\}_{t=1}^{\tau-1})$ made by agents prior to time τ. If it is not player i's turn to move in period t then $c_i^t=0$. A *strategy* for player i specifies the size of the player's contribution for each history after which it is player i's turn to move.

Players are impatient, and discount both contributions and benefits using an identical

5. The restriction to two players is made for simplicity. Our analysis carries over in a straightforward manner to the case of more than two players.

discount factor $\delta < 1$. Let T be the first time at which the total contributions reaches K. If the project is not completed then $T = \infty$. An outcome of the game is defined by a triple $(T, \{c_1^t\}_{t=1}^T, \{c_2^t\}_{t=1}^T)$. For $i = 1, 2$, player i's payoff is given by

$$U_i(T, \{c_1^t\}_{t=1}^T, \{c_2^t\}_{t=1}^T) = \delta^{T-1} V - \sum_{t=1}^T \delta^{t-1} W(c_i^t), \tag{2.1}$$

where $W(\cdot)$, which measures the disutility from the contribution, is increasing, strictly convex and satisfies $W(0) = 0$. This particular payoff function captures the idea that agents bear the cost of each contribution at the time the contribution is made. However, benefits are only obtained when enough contributions have been made, i.e. in period T. Note that our specification treats costs and benefits symmetrically, namely both are discounted. Agents prefer to delay their contributions but are impatient to receive benefits.[6]

The set of Nash equilibria of the above game is very large. Below we seek a subgame-perfect equilibrium to the game (see Selten (1975)). In a subgame-perfect equilibrium a player's strategy is optimal after every history, not only at the beginning of the game. Hereafter the term "equilibrium" refers to a subgame-perfect equilibrium.

Remark. Note that in our model only one agent contributes to the project in any given period. This seems to be appropriate if simultaneous contributions are technically impossible. However, it may be that the technology allows both players to contribute in every period. In such cases one might argue that our model "forces" each of the players to be idle for half of the time. An alternative model, which captures the possibility of simultaneous contributions and at the same time keeps the complete and perfect information structure of our game is as follows. Suppose players can contribute a positive amount in every period but can only change their rate of contribution every other period. Also assume that player 1 can change his rate every even period while player 2 every odd period. The only difference between this model and the one we analyse here is in the way payoffs are now defined. Specifically, the cost of choosing a rate of investment c is now $W(c)(1+\delta)$ instead of just $W(c)$, and the total contribution implied by this choice is $2c$. The analysis of this model is completely analogous to the one we present below for the game with alternating contributions, and it leads to similar results.

3. THE MAIN RESULT

We now characterize the equilibrium of the above game. To do this we will define a sequence of critical contribution levels, denoted by $\{R_n\}_{n=1}^{\infty}$. This sequence will describe the contributions of the agents along the equilibrium path, with R_n being the level of contribution n rounds from the completion of the project. Specifically, if the project is completed in the equilibrium then (after one step in which the remaining contribution is equated to the closest partial sum of the R_n's), the players alternate making contributions along the sequence of R_n's until the project is completed. The project is not completed in equilibrium if such a process does not lead to a total contribution of exceeding K.

As will become clear, the relevant information in any history in our game is summarized by a simple state variable describing the total amount of contribution still required for the completion of the project. We denote this variable by X. After a history $(\{c_1^t\}_{t=1}^{\tau}, \{c_2^t\}_{t=1}^{\tau})$ we have

$$X = K - \sum_{t=1}^{\tau} c_1^t - \sum_{t=1}^{\tau} c_2^t.$$

6. It is straightforward to adapt our analysis of Section 3 to the case in which only benefits are discounted, so that the payoff function is given by $\delta^{T-1} V - \sum_{i=1}^T W(c_i^t)$, as well as for the case of fixed waiting costs, where the payoff function is given by $V - \mu T - \sum_{i=1}^T W(c_i^t)$.

For the rest of this section it is also useful to define the partial sum $S_n = \sum_{q=1}^{n} R_q$. Also let $S_0 = 0$. We are now ready to construct the sequence of critical contribution levels. Simultaneously, we will define two functions which will describe the equilibrium payoffs of the first and second player to move as a function of X.

Step 1. Define R_1 as the maximum amount that a player is willing to contribute now if by doing so he completes the project, while if he contributes zero then the project is completed one period later by the other player. Formally, R_1 is the solution to

$$V - W(R_1) = \delta V. \tag{3.1}$$

It is easy to see that if in a subgame $X < R_1$, then it is a dominant strategy for the player whose turn it is to move to complete the project in his turn.

Now, for any $0 \leqq X \leqq R_1$, let

$$U_a^*(X) = V - W(X) \quad \text{and} \quad U_b^*(X) = V. \tag{3.2}$$

Clearly, $U_a^*(X)$ and $U_b^*(X)$ are the payoffs of the first and second players to move respectively if the size of the project is X and the first player completes the project in his turn.

Step 2. Let R_2 be the contribution level that makes a player indifferent between

(i) contributing R_2 under the assumption that the project will be completed in the next period by the other player, and
(ii) contributing zero now and completing the project in two periods by contributing R_1 then (with the rest contributed by the other player in the next move).

Formally, R_2 is defined by

$$\delta V - W(R_2) = \delta^2 V - \delta^2 W(R_1), \tag{3.3}$$

which, using (3.2), can be written as

$$\delta U_b^*(S_1) - W(R_2) = \delta^2 U_a^*(S_1). \tag{3.4}$$

The left-hand side describes the payoff of the player whose turn it is to move if he contributes R_2 now and becomes the second mover starting in the next period (assuming there is $S_1 = R_1$ left to contribute then). The right-hand side describes his payoff if he contributes zero and then he obtains the payoff of the first mover in period two (again assuming that there is $S_1 = R_1$ left to contribute then).

Step 1 above defines $U_a^*(X)$ and $U_b^*(X)$ for any $0 \leqq X \leqq S_1$. We can now extend this definition. For $S_1 < X \leqq S_2$, define

$$U_a^*(X) = \delta U_b^*(R_1) - W(X - R_1) \quad \text{and} \quad U_b^*(X) = \delta U_a^*(R_1). \tag{3.5}$$

These payoff functions are calculated under the conjecture (which will be verified in equilibrium) that if $S_1 < X \leqq S_2$ then the first mover contributes enough so that after his move the required contribution is R_1, and the second mover completes the project in his turn.

Step n. Define R_n recursively as the amount that makes player i indifferent between the following:

(i) contributing R_n now, and obtaining $U_b^*(S_{n-1})$ in the next period, and
(ii) contributing zero now and obtaining $U_a^*(S_{n-1})$ in two periods.

Thus, R_n solves

$$\delta U_b^*(S_{n-1}) - W(R_n) = \delta^2 U_a^*(S_{n-1}). \tag{3.6}$$

Finally, for every $S_{n-1} < X \leqq S_n$, we define

$$U_a^*(X) = \delta U_b^*(S_{n-1}) - W(X - S_{n-1}) \quad \text{and} \quad U_b^*(X) = \delta U_a^*(S_{n-1}). \tag{3.7}$$

The following is straightforward to show.

Lemma 3.1.

(i) *For* $n \geqq 1$,

$$U_a^*(S_n) = \delta U_b^*(S_n). \tag{3.8}$$

(ii) *For* $n \geqq 2$,

$$W(R_n) = \delta^{2n-3} V(1 - \delta^2). \tag{3.9}$$

We are now ready to state and prove the main result of this section, which describes the (essentially) unique equilibrium of our game.

Proposition 3.1. *Let* $S_\infty = \sum_{q=1}^{\infty} R_q$.

(i) *Suppose* $S_\infty > K$.

(ia) *If there exists* $N < \infty$ *such that* $S_{N-1} < K < S_N$, *then the unique equilibrium path is: player* 1 *contributes* $K - S_{N-1}$ *in period* 1, *and for* $1 < t \leqq N$, *the amount contributed in period* t *is* R_{N-t+1}. *Thus, the project is completed in* N *rounds. Player* 1'*s equilibrium payoff is* $U_a^*(K)$ *and player* 2'*s equilibrium payoff is* $U_b^*(K)$.

(ib) *If there exists* $N < \infty$ *such that* $S_N = K$ *then there are two equilibrium paths. In addition to the path described in* (*ia*), *there is an equilibrium path in which player* 1 *contributes zero in period* 1 *and from period* 2 *on the path is the same as in* (*iia*), *with player* 2 *making the first positive contribution. In the second equilibrium player* 1'*s payoff is* $\delta U_b^*(K) = U_a^*(K)$, *and player* 2'*s payoff is* $\delta U_a^*(K)$.

(ii) *If* $S_\infty \leqq K$, *then the unique equilibrium path is* $c_i^t = 0$ *for all* i *and* t. *In this case the project is not completed and players' payoffs are zero.*

Proof. Recall that for a given history, X is the total amount of contributions still required to complete the project. For a given history denote by a the player whose turn it is to move and by b the other player.

We first prove part (i). For a given history let n satisfy $X \leqq S_n$. We will prove by induction on n that the proposition holds for every subgame in which a total of $X \leqq S_n$ is required to complete the project. The result will follow immediately.

Suppose $n = 1$, i.e. $0 < X \leqq S_1 = R_1$. If $X < R_1$ then player a has the choice between $V - W(X)$ or at most δV. From the definition of R_1, player a strictly prefers the former, i.e. to complete the project at once. Thus, player a's equilibrium payoff is $U_a^*(X)$ and player b's equilibrium payoff is $U_b^*(X)$. Using again the definition of R_1 it follows that if $X = R_1$ then there is exactly one additional equilibrium path, i.e. player a contributes zero now and player b completes the project in the next period. The equilibrium payoff in this case follows easily.

Assume that the proposition holds for every X such that $0 < X \leqq S_{n-1}$. We will show that it also must hold for every X such that $0 < X \leqq S_n$. We first discuss the case $X < S_n$.

Claim 1. *If $X > S_{n-1}$, then there is no equilibrium in which player a contributes $X - S_q$ where $q < n-1$.*

Proof. From the induction hypothesis we know that for a small enough $\varepsilon > 0$, player a can obtain the payoff $\delta^2 U_a^*(S_q) - W(X - S_{q+1} + \varepsilon)$ by contributing $X - S_{q+1} + \varepsilon$. From (3.7) this is equal to $\delta U_b^*(S_{q+1}) - W(X - S_{q+1} + \varepsilon)$. We will show that this is strictly higher than $\delta U_b^*(S_q) - W(X - S_q)$, which is player a's payoff if he contributes $X - S_q$. To see this note that strict convexity of $W(\cdot)$ and the fact that $W(0) = 0$ imply that

$$W(R_{q+1}) + W(X - S_{q+1}) < W(X - S_q). \tag{3.10}$$

It follows that

$$\delta U_b^*(S_q) - W(X - S_{q+1}) - W(R_{q+1}) > \delta U_b^*(S_q) - W(X - S_q). \tag{3.11}$$

But from the definition of R_q we have $\delta U_b^*(S_q) - W(R_{q+1}) = U_a^*(S_{q+1})$. This and part (i) of Lemma 3.1 implies that for a small enough $\varepsilon > 0$,

$$\delta U_b^*(S_{q+1}) - W(X - S_{q+1} + \varepsilon) > \delta U_b^*(S_q) - W(X - S_q), \tag{3.12}$$

which proves the claim. ||

Claim 2. *If $X > S_{n-1}$, then there is no equilibrium in which player a contributes strictly more than $X - S_{n-1}$.*

Proof. Suppose by contradiction that there exists an equilibrium in which player a reduces X to $X' < S_{n-1}$. From Claim 1, there is no q such that $X' = S_q$. By the induction hypothesis, $U_b^*(\cdot)$ is constant on each interval (S_{q-1}, S_q) for $q \leqq n-1$. Since the cost function is increasing, there exists a small $\varepsilon > 0$ such that player a can do strictly better if he reduces his contribution by ε. (The project will be completed at the same time, but the player will contribute less.) ||

Claim 3. *Suppose $S_{n-1} < X < S_n$. Then there is no equilibrium in which player a contributes strictly less than $X - S_{n-1}$.*

Proof. From the induction hypothesis, player a can guarantee himself a payoff arbitrarily close to $\delta U_b^*(S_{n-1}) - W(X - S_{n-1})$. If player a contributes strictly less than $X - S_{n-1}$ then it follows from the previous claims that his highest possible payoff is $\delta^2 U_a^*(S_{n-1})$. But by the definition of R_n,

$$\delta U_b^*(S_{n-1}) - W(X - S_{n-1}) > \delta U_b^*(S_{n-1}) - W(R_n) = \delta^2 U_a^*(S_{n-1}). \quad ||$$

Claim 4. *If in equilibrium player a contributes $X - S_{n-1}$, then player b contributes R_{n-1} in his next move.*

Proof. From the induction hypothesis, in the subgame following player a's move there are two equilibria. In one equilibrium player b contributes R_{n-1} and in the other he contributes zero. The latter, however, cannot be part of the overall equilibrium, since player a would strictly prefer to deviate from the proposed strategy and contribute $X - S_{n-1} + \varepsilon$ for a small enough $\varepsilon > 0$. ||

The four claims above imply that for $X < S_n$ there is at most one equilibrium, namely the one described in the proposition.

Finally, consider the case $X = S_n$. Claims 1 and 2 imply that in any equilibrium player a contributes either R_n or zero. Moreover, from an argument similar to the proof of claim 4, if player a contributes zero in this period then player b contributes R_n in the following period.

To complete the proof of part (i) one can show that the following strategy for each player constitutes an equilibrium.

- If it is your turn to move and $S_{q-1} < X \leqq S_q$ then contribute $X - S_{q-1}$.

It follows that player a's equilibrium payoff is $U_a^*(X)$ and player b's equilibrium payoff is $U_b^*(X)$.

We now prove part (ii), where $S_\infty \leqq K$. First we show that there is no equilibrium in which the project is completed. Suppose that there exists an equilibrium in which the project is completed after N rounds. Clearly, $K > R_1$. Thus, from the proof of part (i), the player who completes the project must contribute exactly R_1 in round N. Similarly, it must be that exactly R_2 is contributed in round $N-1$, and thus no more than R_N is contributed in the first round. But for every N, $S_N < K$, which is a contradiction to the assumption that the project is completed in N rounds. Thus, the only candidate for an equilibrium path is that $c_i^t = 0$ for every i and t. A strategy for each player that supports this as an equilibrium path is given by

- If it is your turn and there exists n such that $S_{n-1} < X \leqq S_n$, then contribute $X - S_{n-1}$; otherwise contribute zero.

To see that this is indeed an equilibrium strategy note that if, after some history, $X < S_\infty$ then from the proof of part (i) the above is an equilibrium continuation of the game. When $X \geqq S_\infty$ it is clear from the proof of part (i) and from the argument above that the player whose turn it is to move will either contribute zero or $X - S_q$ for some q. We now show that contributing zero leads to a higher payoff to the player, hence it is his best response. If player a (the player whose turn it is to move) contributes $X - S_q$ then his payoff will be $\delta U_b^*(S_q) - W(X - S_q)$. Now, since $X > S_\infty$, $X - S_q > \sum_{n=q+1}^{\infty} R_n$. Also it can be shown that $U_b^*(S_q) = \delta^{2q-2} V$. Thus,

$$\delta U_b^*(S_q) - W(X - S_q) < \delta^{2q-1} V - W\left(\textstyle\sum_{n=q+1}^{\infty} R_n\right). \tag{3.13}$$

Convexity of $W(\cdot)$ implies that the right-hand side of (3.13) is strictly smaller than $\delta^{2q-1} V - \sum_{n=q+1}^{\infty} W(R_n)$, which, using (3.9) and some algebra, can be shown to be equal zero. This completes the proof of the proposition. ||

The above result provides a complete characterization of the equilibrium path—essentially, players take turn contributing R_n's with the last contribution being R_1. Some properties of the equilibrium are immediate. First, it is easy to see that

Lemma 3.2.

(i) *For* $q \geqq 2$, R_q *is decreasing in* q.
(ii) $R_2 > R_1$ *if and only if* $(1-\delta)(1-\delta^2-\delta) < 0$, *i.e. if and only if* $\delta > 0{\cdot}618034$.

Thus, in equilibrium, if δ is large enough then players' contribution levels are increasing over time, except that the last contribution is smaller than the one before last. If δ is small then contribution levels are always increasing over time.

Another result that follows directly from Proposition 3.1 is

Corollary 3.1. *Player 2's equilibrium payoff if the project is completed in N periods is* $U_b^*(K) = \delta^{2N-2}V$. *Player* 1'*s payoff,* $U_a^*(K)$, *satisfies* $U_b^*(K)\delta \leqq U_a^*(K) \leqq U_b^*(K)/\delta$.[7]

This implies that when $N \geqq 2$ the payoffs of the two players cannot generally be ranked—depending on the parameters of the model, it is possible that either player 1 or player 2 will obtain a higher payoff relative to the other player. Note that when δ approaches 1, the difference between the payoffs of the two players vanishes.

4. THE INEFFICIENCY OF THE EQUILIBRIUM

In this section we describe the underlying parameters for which the project is completed in equilibrium. We derive a necessary condition for the completion of the project and show that the equilibrium outcome of the contribution game is inefficient in the sense that projects which are socially desirable may not be completed in equilibrium. As described by Schelling in the Red Cross example, the basic intuition is that, because of the lack of a commitment technology, each agent holds back his contributions in order to verify that the other player would also contribute his share.

We first analyse the limit case of our model in which the cost function $W(\cdot)$ is linear. In addition to being interesting in its own right, the linear case provides key insights into the model with a strictly convex cost function. A linear cost function is given by $W(c) = bc$ with $b > 0$. Since this function is not strictly convex, our results so far do not cover this case. However, it is easy to see that Proposition 3.1 describes an equilibrium path for the model with this cost function. (Uniqueness does not hold in this case, however; see below.) As the next result shows, the expressions for the critical values and the condition for the completion of the project are much simpler in this case.

Lemma 4.1. *If* $W(c) = bc$ *then* $S_n = V(1-\delta^{2n-1})/b$, *and* $S_\infty = V/b$. *Thus, the project is completed in the equilibria of Proposition* 3.1 *if and only if* $V > W(K) = bK$. *Player* 1'*s equilibrium payoff if the project is completed is* $U_a^*(K) = V - bK = V - W(K)$.

The above lemma says that in the linear case *a necessary and sufficient condition for the completion of the project in our equilibrium is that each player would complete the project (immediately) if he was the only player.* Moreover, player 1's equilibrium payoff in the linear case is exactly the same as it would have been if he is the only player! Although the equilibrium path is no longer unique in the case of a linear cost function, it is easy to see that such a property holds for any equilibrium.[8] Intuitively, as already discussed, the inability to commit results in equilibria in which agents hold back and delay their contributions. The delay in completion exactly offsets the gains from cooperation, so that player 1 is indifferent between the equilibrium outcome and the outcome he would obtain if he was the only player. This is an extreme form of inefficiency.

7. The derivation of $U_b^*(K)$ is straightforward given the definition of the sequence R_n. The properties of $U_a^*(K)$ follow from the fact that $U_a^*(K) = \delta^{\max(0,2N-3)}V - W(K - S_{N-1})$.

8. To see how the linear case differs from the strictly convex case note that if $W(\cdot)$ is linear, then a player is indifferent between (i) completing the project and (ii) contributing $K - R_1$ and having the other player complete the project in the next period. Thus, if, for example, $S_1 < K \leqq S_2$ then there are three equilibria. In one of them player 1 completes the project in his turn; in the second player 1 contributes $K - R_1$ and player 2 completes the project, and in the third player 1 contributes zero and player 2 completes the project in his turn. However, in all of the equilibria the project is completed if and only if $V > W(K)$, i.e. if and only if the project would be completed by one player if he was the only one. Also note that in an equilibrium there exists a player whose payoff is at most $V - W(K)$, which is his payoff if he is the only player.

In the general case of a strictly convex cost function the situation is somewhat more complicated. Clearly, if $V > W(K)$ then the project must be completed in equilibrium. However, this condition is not necessary for the completion of the project. (Indeed, this condition is not necessary for the completion of the project by one player if he was the only player, since with a convex cost function delay may arise even in the single-person problem.) The next result provides a necessary condition for the completion of the project in the case of a general convex cost function.

Proposition 4.1. *If $V \leqq W'(0)K$ then the project is not completed in equilibrium.*

Proof. Observe that if $\bar{W}(\cdot)$ and $\underline{W}(\cdot)$ are strictly convex and increasing functions satisfying $\bar{W}(0) = \underline{W}(0) = 0$ and $\bar{W}(c) > \underline{W}(c)$ for all $c > 0$, then (i) if the project is completed when the cost function (in (2.1)) is $\bar{W}(\cdot)$ then it is also completed if the cost function is $\underline{W}(\cdot)$, and (ii) in general, the length of time until the project is completed is larger if the cost function is $\bar{W}(\cdot)$ than it is if the cost function is $\underline{W}(\cdot)$. To see this note that under the given assumptions for every n, the value of R_n when $W(\cdot) = \underline{W}(\cdot)$ is larger than the corresponding value of R_n when $W(\cdot) = \bar{W}(\cdot)$.

Since $W(\cdot)$ is convex, $W(c) \geqq W'(0)x$ for all $c > 0$. The proposition now follows from the above observation together with Lemma 4.1. ||

Note that the necessary condition for completion provided in Proposition 4.1 does not depend on the discount factor δ. The proposition implies that in general the equilibrium outcome of the contribution game is inefficient in the sense that projects which are socially desirable may not be completed in equilibrium. As a specific example, suppose $V = K = 1$ and $W(c) = c + c^2$. Then $1 = V > W(K/2) = 3/4$, so that for δ close enough to 1 players obtain a strictly positive payoff if each one of them incurs half of the cost of the project in his turn. However, since $V = W'(0)K$ we know from Proposition 4.1 that the project is not completed in the equilibrium of the contribution game.

It is analytically difficult to characterize the precise set of projects which are completed in the equilibrium of the game, or the set of socially desirable projects, i.e. those which are completed in the efficient, first best world. In Appendix A we provide some such characterizations for the limit in which the time between moves vanishes. We show that, at least for the case of power cost functions, the set of projects completed in the equilibrium of the game is strictly smaller than the set of socially desirable projects.

5. THE SUBSCRIPTION GAME

The contribution game addresses a situation in which commitments and enforceable contracts are not available. In this game the cost of the contributions is borne and sunk when they are made. To understand the role of the lack of commitment in the analysis and particularly in the inefficiency result obtained above, we now analyse an alternative procedure, which we call the *subscription game.* In this game the cost of the contributions is only borne when and if enough contributions are "pledged" to complete the project. In other words, agents are able to make certain conditional commitments to contribute in the future.

The game proceeds as follows. Player 1 starts by subscribing, i.e. committing to contribute, a certain amount to the project if and when the total amount of subscriptions exceeds the total cost K. Then it is player 2's turn, followed by player 1 again, etc. In every round a player can only add to the amount that has already been committed or subscribed by him previously. Let C_i be the total subscription of player i at some stage of the game. Let T be the first time such that $C_1 + C_2 \geqq K$. An outcome of the game (sufficient to compute payoffs) is defined by (T, C_1, C_2). For simplicity, we assume that the players' payoff are given by

$$U_i(T, C_1, C_2) = \delta^T(V - C_i), \tag{5.1}$$

for $i = 1, 2$.

Recall that if the cost function in the contribution game is given by $W(c) = c$ then a necessary condition for the completion of the project in the equilibrium is that $V > K$, i.e. that each player would complete the project if he is the only player. More generally, we have shown that the outcome of the contribution game is inefficient in the sense that socially desirable projects may not be completed in equilibrium. In contrast, we will show that all socially desirable project are completed in the equilibrium of the subscription game. That is, the project is completed whenever $K < 2V$. Moreover, there is only one round of subscriptions in the (essentially) unique subgame-perfect equilibrium path.

Proposition 5.1.

(i) *If $K > 2V$ then the unique equilibrium path of the subscription game is that each player subscribes zero and the project is not completed.*[9]

(ii) *If $V(1-\delta) < K < 2V$ then there is a unique equilibrium path to the subscription game, in which player* 1 *subscribes in the first move*

$$C_1^* = \frac{K - V(1-\delta)}{1+\delta}, \tag{5.2}$$

and player 2 *in his turn completes the project by subscribing*

$$C_2^* = K - C_1^* = \frac{\delta K + V(1-\delta)}{1+\delta}. \tag{5.3}$$

(iii) *If $K < V(1-\delta)$ then the unique equilibrium path of the subscription game is that player* 1 *subscribes K in the first move and completes the project.*

(iv) *If $K = V(1-\delta)$ then the subscription game has exactly two equilibrium paths. In one equilibrium path player* 1 *subscribes K in the first move and completes the project. In the second equilibrium path player* 1 *subscribes zero and player* 2 *subscribes K and completes the project in his turn.*

Proof. See Appendix B. ‖

Note that for the special case $V = K = 1$, we get $C_1^* = \delta/(1+\delta)$ and $C_2^* = 1/(1+\delta)$. That is, C_1^* is exactly what player 2 obtains in the equilibrium of Rubinstein's (1982) bargaining game, while C_2^* is what 1 obtains in that equilibrium. In fact, the subscription game is analogous to a bargaining game in which each player in his turn announces his demand for a share of the pie, and this demand can be reduced (but cannot increase) in subsequent rounds. When the total demand is below one, agreement is reached and the pie is divided accordingly.

9. If $K = 2V$ then in equilibrium either both players subscribe zero and the project is not completed or each one player subscribes V and the project is completed. In the latter case the time in which subscriptions are made is not unique. However, in all of these equilibria players' equilibrium payoffs are zero.

To understand the difference between the contribution game and the subscription game note that, as already discussed, the source of the inefficiency in the equilibrium of the contribution game is that each agent holds back his contribution in any stage to assure that the other agent contributes his share as well. This is because past contributions are sunk and so do not influence the division of the remaining cost among the agents. In contrast, in the subscription game the cost of a particular subscription is borne only upon the completion of the project. Thus, the more a player has already pledged, the less is the completion of the project worth to him and therefore the less incentive he has to reach an early agreement. It is therefore "credible" for a player to subscribe his entire share at once and refuse to increase his contribution later.

APPENDIX A

The limit as the time between moves vanishes.

Suppose the length of time between moves is ρ, and let $\delta(\rho)=\exp(-r\rho)$ be the discount factor per period, where $r>0$. To enable us to change ρ without changing the cost structure, we need to reinterpret our model slightly. We now think of the players as choosing the *rate* at which to invest in each period. If player i chooses to contribute at a rate of c_i^t in period t, then his total contribution in period t is ρc_i^t, and the cost he incurs in period t is $\rho W(c_i^t)$. The analysis of the previous sections corresponds to the case $\rho=1$. If $\rho\neq 1$ the analysis must be modified. In this case R_n, which is now interpreted as a *rate* of contribution, satisfies

$$W(R_1)=\frac{V(1-\delta)}{\rho}=\frac{V(1-e^{-r\rho})}{\rho} \tag{A1}$$

and for $n\geqq 2$,

$$W(R_n)=\frac{\delta^{2n-3}V(1-\delta^2)}{\rho}=\frac{e^{-r\rho(2n-3)}V(1-e^{-2r\rho})}{\rho}. \tag{A2}$$

It is easy to adapt our previous analysis to this interpretation and to verify the validity of the basic results. In particular, the project is completed in the equilibrium of the contribution game if and only if $\sum_{n=1}^{\infty}\rho R_n>K$. This condition can be written as

$$\rho W^{-1}\left(\frac{V(1-e^{-r\rho})}{\rho}\right)+\sum_{n=2}^{\infty}\rho W^{-1}\left(\frac{e^{-r\rho(2n-3)}V(1-e^{-2r\rho})}{\rho}\right)>K. \tag{A3}$$

The next result provides conditions for the completion of the project in the contribution game as the time between moves vanishes.

Proposition A1.

(i) *Suppose*

$$\int_0^{\infty} W^{-1}(2rVe^{-2tr})\,dt>K. \tag{A4}$$

Then there exists $\rho>0$ such that for all $\rho'<\rho$ the project is completed in the equilibrium of the game with period length ρ'.

(ii) *Suppose*

$$\int_0^{\infty} W^{-1}(2rVe^{-2tr})\,dt<K. \tag{A5}$$

Then there exists $\rho>0$ such that for all $\rho'<\rho$ the project is not completed in the equilibrium of the game with period length ρ'.

Proof. Substituting t/ρ for n in the left-hand side of (A3) and taking the limit as $\rho \to 0$, we have, using L'Hôpital's rule,

$$\lim_{\rho\to 0} \rho W^{-1}\left(\frac{V(1-e^{-r\rho})}{\rho}\right) = 0, \tag{A6}$$

and

$$\lim_{\rho\to 0}\left(\sum_{n=2}^{\infty} \rho W^{-1}\left(\frac{e^{-r\rho(2n-3)}V(1-e^{-2r\rho})}{\rho}\right)\right) = \int_0^{\infty} W^{-1}(2rVe^{-2tr})dt. \tag{A7}$$

(Convergence of the sum to the Riemann integral follows because the integrand is decreasing.) From the continuity of the left-hand side of (A3) in ρ, we have that if (A4) holds then for small enough ρ, (A3) holds. Similarly, if (A5) holds, then the inequality in (A3) is strictly reversed. ||

We wish to compare the conditions obtained above for the contribution game with those that would obtain in the first-best, efficient world. In the first-best world the two players coordinate their actions, so that the outcome corresponds to a single agent whose benefit upon completion of the project is $2V$, the total benefit to both players. The comparison between the game and the efficient outcome is complex in general. However, we will be able to obtain considerable insight from analysing the special case where $W(\cdot)$ is a power function, i.e. $W(c) = c^{\alpha}$ with $\alpha > 1$. From Proposition A1, whether the project is completed in equilibrium as $\rho \to 0$ depends on whether K is smaller or larger than

$$\int_0^{\infty} W^{-1}(2rVe^{-2tr})dt = \frac{\alpha}{2r}(2rV)^{1/\alpha}. \tag{A8}$$

We now find the set of parameters for which the project is completed in the efficient, first-best outcome for the case $W(c) = c^{\alpha}$. To do this we need to consider the problem of a single agent who chooses in each period a rate of investment in a project that is worth $2V$ upon completion. Since the discrete-time optimization problem in this case is analytically complex, we focus on the continuous-time version of the problem, which will be used to characterize the limit as $\rho \to 0$. The continuous-time maximization problem, assuming the project is completed, can be written as[10]

$$\max_{T,c_t}\left\{2Ve^{-rT} - \int_0^T e^{-rt}(c_t)^{\alpha}dt\right\}, \tag{A9}$$

subject to

$$\int_0^T c_t dt = K. \tag{A10}$$

We have

Proposition A2.

(i) *Suppose*

$$(2V)^{1/\alpha}\left(\frac{\alpha-1}{r}\right)^{1-1/\alpha} > K. \tag{A11}$$

Then there exists $\rho > 0$ such that for all $\rho' < \rho$ the project is completed in the efficient outcome where the period length is ρ'.

(ii) *Suppose*

$$(2V)^{1/\alpha}\left(\frac{\alpha-1}{r}\right)^{1-1/\alpha} < K. \tag{A12}$$

Then there exists $\rho > 0$ such that for all $\rho' < \rho$ the project is not completed in the efficient outcome where the period length is ρ'.

10. More precisely, the admissible controls in the continuous-time problem are a time $T \in [0, \infty]$ and a non-negative function c_t describing the rate of contribution at time t such that c_t is integrable on any interval $[0, T]$, and such that $\int_0^T c_t$ is a continuous function of T.

Proof. Consider the optimization problem in equations (A9) and (A10). The Lagrangian for this problem can be written as:

$$L = 2Ve^{-rT} - \int_0^T e^{-rt}(c_t)^\alpha dt - \lambda\left(K - \int_0^T c_t dt\right). \tag{A13}$$

From the first-order condition to the maximization of (A13) we get:

$$e^{-rt}\alpha c_t^{\alpha-1} = \lambda, \tag{A14}$$

which implies

$$\alpha c_0^{\alpha-1} = \lambda.$$

This allows us to express c_t as a function of c_0:

$$c_t = e^{\left(\frac{rt}{\alpha-1}\right)} c_0. \tag{A15}$$

Substituting (A15) into the constraint (A10) and rearranging gives

$$e^{-rT} = \left(\frac{Kr + c_0(\alpha-1)}{c_0(\alpha-1)}\right)^{1-\alpha}. \tag{A16}$$

Using (A15) and the constraint (A10) we can simplify the integral term in the objective function as follows

$$\int_0^T e^{\left(\frac{rt\alpha}{\alpha-1}\right)} dt = c_0^{\alpha-1}\int_0^T c_t dt = c_0^{\alpha-1}K. \tag{A17}$$

We can now write the maximization problem in equations (A9)-(A10) as

$$\max_{c_0}\left\{2V\left(\frac{Kr + c_0(\alpha-1)}{c_0(\alpha-1)}\right)^{1-\alpha} - c_0^{\alpha-1}K\right\}. \tag{A18}$$

It can be shown that, as is intuitively clear, the objective function above is decreasing in K. The project is completed in equilibrium if and only if the value of the objective function at the optimum is positive. Thus, for a given level of c_0 the set of values K for which the project is completed includes all K lower than that which equates the objective function with zero. That is, for a given c_0, the maximal K such that the project is completed satisfies

$$2V\left(\frac{Kr + c_0(\alpha-1)}{\alpha-1}\right)^{1-\alpha} = K. \tag{A19}$$

Viewing the K that solves this equation as a function of c_0, it is straightforward to show that this K is decreasing in c_0. (That is, the higher is the initial contribution rate c_0 the lower is the highest cost of the project for which the project is completed by starting with a contribution rate c_0 and proceeding optimally.) To find the maximal K for which there exists an initial contribution rate c_0 such that completing the project yields positive payoffs (i.e. such that the maximized objective function is positive) we take the limit as $c_0 \to 0$ in the expression in (A18) and solve for K. To complete the proof we note that the optimal value of the discrete-time versions of the problem converges to the optimal value of the continuous-time problem. More specifically, if the optimal value for the continuous-time problem is positive then for sufficiently small ρ the optimal value of the discrete-time problem with period length ρ is also positive, and if the optimal value of the continuous-time problem is negative then for sufficiently small ρ the optimal value of the discrete-time problem with period length ρ is negative. Proving these claims is straightforward. ||

A comparison of the expression on the right-hand side of (A6) with that on the left-hand side of (A9), and Propositions A1 and A2 lead to the following conclusion.

Corollary A1. *As ρ vanishes, for all $\alpha > 1$, if $\alpha \neq 2$, then the equilibrium of the contribution game is inefficient. That is, the set of projects that are completed in the first-best world is strictly larger than the set of projects that are completed in the equilibrium of the game.*

APPENDIX B

Proof of Proposition 5.1. If $k>2V$ then it is clear that in order for the project to be completed at least one player must contribute more than V, which can never be an equilibrium outcome. Clearly, the unique equilibrium path in this case is that agents subscribe zero and the project is not completed. This proves (i).

Next we take the case in which $K<V(1-\delta)$, i.e. case (iii). Suppose that player 1 has pledged C_1 in total, player 2 has pledged C_2 (where $C_1+C_2<K$), and it is the turn of player 1 to pledge. We claim that player 1 will pledge what remains to complete the project immediately. By so doing, his payoff (in current value terms) is $V-K+C_2$. (Since this is positive, player 1 will certainly complete the project if the alternative is that the project is never completed.) If he fails to complete the project, then the best he can hope for is that the project is completed immediately by player 2, giving him a current value payoff of $\delta(V-C_1)$. But for any C_1, $C_2 \geqq 0$, in the case $K<V(1-\delta)$ the former strictly exceeds the latter. Hence in this case player 1 completes the project immediately in all cases, as does player 2. This establishes (iii).

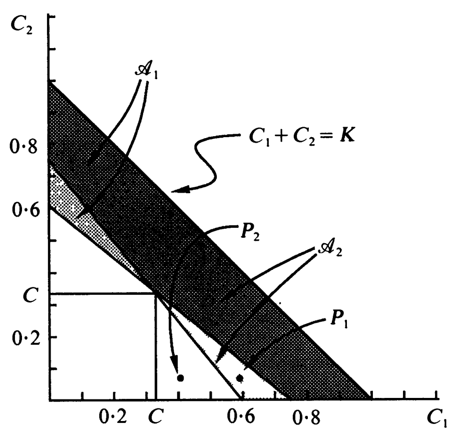

FIGURE 1

Now assume that $V(1-\delta)<K<2V$, i.e. case (ii). For the remainder of the proof, we use indices a and b for two players, where player a will usually be the player who subscribes next.

Define $A(C)$ by

$$A(C)=(1-\delta)(V-C). \tag{B1}$$

We claim that if a player has already subscribed C, if it is his turn to contribute, and if it takes less than $A(C)$ to complete the project, then the player will complete the project immediately in any subgame-perfect equilibrium. The logic is a simple adaptation of the argument in the preceding paragraph and is therefore left to the reader. In other words, if we let $\mathscr{A}_a=\{(C_1, C_2) \mid C_1+C_2>K-A(C_a)\}$, then from any point in the set $\mathscr{A}_a$, if it is a's turn to contribute, he will complete the project immediately.

Figure 1 shows the sets $\mathscr{A}_1$ and $\mathscr{A}_2$, as well as other constructs of the equilibrium for the case $K=1$, $V=2$, and $\delta=0{\cdot}8$. (This is one of two typical pictures which differ according to whether $K<V$ or $K>V$.) Note that for all points above the line $C_1+C_2=K$ pledges are sufficient to complete the project. Thus, we are concerned only with points in the region $C_1+C_2<K$ (and both C_1 and C_2 nonnegative).

Denote by $\bar{\mathscr{A}}_a$ the closure of $\mathscr{A}_a$. Let $\hat{C}$ be the solution of the equation $2C=K-A(C)$, i.e. $\hat{C}$ is the intersection of the two lines $A(C_2)=K-C_1-C_2$ and $A(C_1)=K-C_1-C_2$. By simple algebra

$$\hat{C}=\frac{K-V(1-\delta)}{1+\delta}. \tag{B2}$$

Note that in case (ii), $0<\hat{C}<K/2$.

Now suppose that the game has evolved to a point where it is player a's turn to subscribe, $(C_1, C_2) \notin \bar{\mathscr{A}}_a$, and $(C_1, C_2) \in \mathscr{A}_b$. (For example, for $a=1$ we are at a point like P_1 in Figure 1.) Player a knows that if he subscribes zero in his turn, or if he subscribes any further amount still insufficient to complete the project, then in any subgame perfect equilibrium player b will complete the project in the next turn. It is then straightforward to show that player a's best action at this point is to subscribe zero. (Essentially, this is the reverse of the inequality that defines the line $A(C_a)$.)

Suppose that the game has evolved to a point where it is player a's turn to pledge, $C_a > \hat{C}$, and $(C_1, C_2) \notin \bar{\mathcal{A}}_1 \cup \bar{\mathcal{A}}_2$. (For example, for $a = 1$ we are at a point like P_2 in Figure 1.) We claim that in this case player a subscribes zero and the other player (player b) completes the project in his turn. The fact that a subscribes zero will follow from logic similar to that given above, as soon as we show that player b will complete the project from a point such as P_2 (for $b = 2$). We now prove this assertion.

Fix $C_b < \hat{C}$ and let C_a^* be the infimum over all values of C_a such that in all subgame-perfect equilibria player b completes the project if it is his turn to subscribe and the position of the game is any (C'_a, C'_b) with $C'_a \geqq C_a$ and $C'_b \geqq C_b$. Essentially, our claim that C_a^* (which could, in principle, depend on the fixed C_b) is equal to $\hat{C}$. We know from previous steps that $C_a^* \leqq K - C_b - A(C_b)$. Suppose that $C_a^* > \hat{C}$. Then for all $\varepsilon > 0$ there is some C'_a such that $C_a^* - \varepsilon < C'_a \leqq C_a^*$, some C'_b such that $\hat{C} > C'_b > C_b$ and some subgame-perfect equilibrium continuation beginning from (C'_a, C'_b) in which it is b's turn to move where b does not complete the project. We will derive a contradiction by choosing ε sufficiently small. To begin, choose ε sufficiently small so that $C_a^* - \varepsilon > \hat{C}$.

Fix this subgame-perfect equilibrium continuation, and let c_b be the immediate subscription of player b. From the definition of C_a^*, we know that player a can contribute anything above ε, in particular 2ε and be assured that the project will be completed by b in his subsequent turn. Hence player a is assured of a payoff (measured from when he moves) of $\delta(V - C_1 - 2\varepsilon)$. Of course, player a might complete the project in his turn, which will give him the payoff $V - C'_b - c_b$. There are two cases to consider:

(1) Suppose player a does not complete the project in his turn in this equilibrium continuation. Then as long as

$$\varepsilon < \frac{(1-\delta)(V - C_a^*)}{2}, \tag{B3}$$

player a is sure to subscribe the smallest amount which causes b to complete the project, and this subscription by a is less than 2ε. To see this, note that the alternative is to subscribe some amount which causes b not to complete the project. But this results in a (present value) payoff of $\delta(V - C'_a)$ at most, while pledging 2ε nets $(V - C'_a - 2\varepsilon)$. Since $C'_a \leqq C_a^*$, the latter is better.

It follows that player b knows that a will pledge at most 2ε in his turn, and the project will be completed in b's subsequent opportunity to pledge. Hence b nets

$$\delta^2(V - K + C'_a + 2\varepsilon)$$

at most, as against $V - K + C'_a$ if b completes the project immediately. A comparison of these two terms shows that if

$$\varepsilon < \frac{(1-\delta^2)(V - K + C'_a)}{\delta^2}, \tag{B4}$$

then the latter is strictly greater. As $C'_a > \hat{C}$ by assumption (and $2V > K$), the right-hand side of the inequality just given is strictly positive.

(2) Suppose player a does complete the project in his turn in this equilibrium continuation. Note that player a retains the option of subscribing 2ε and having player b complete the project subsequently, so in order that player a completes the project it is necessary that

$$\delta(V - C'_a - 2\varepsilon) \leqq (V - K + C'_b + c_b). \tag{B5}$$

Rewrite this inequality as

$$V - C'_b - c_b \leqq (2-\delta)V - K + \delta C'_a + 2\delta^2\varepsilon, \tag{B6}$$

or

$$\delta(V - C'_b - c_b) \leqq V - K + C'_a - (1-\delta)^2 V + (1-\delta)K - (1-\delta^2)C'_a + 2\delta\varepsilon. \tag{B7}$$

Since $C'_a > \hat{C}$, we can pick ε sufficiently small so that $K - V(1-\delta) - (1+\delta)C'_a + 2\delta^2\varepsilon/(1-\delta) < 0$, and thus from the previous inequality

$$\delta(V - C'_b - c_b) < V - K + C'_a. \tag{B8}$$

But this says that player b would rather complete the project immediately than wait one round (after a further pledge of c_b) for a to complete the project.

Now consider a point (C_1, C_2) such that $C_1 < \hat{C}$ and $C_2 < \hat{C}$. We claim that player a (the player whose turn it is to subscribe) will not choose to subscribe strictly more than $\hat{C} - C_a$ along any subgame equilibrium path. To see this, note that a, by subscribing any amount more than this, will get b to complete the project immediately in his turn, giving him a payoff of $\delta(V-\hat{C}-\varepsilon)$ for arbitrarily small ε. We leave it to the reader to show that since $C_a<\hat{C}$ this is better than completing the project immediately. It follows that contributing more (but not enough to complete the project immediately) will only delay completion and increase a's contribution.

Finally, we show that there is no subgame-perfect equilibrium in which player a subscribes less than $\hat{C}$ in total. Suppose by contradiction that he subscribes less in some equilibrium. Then in this equilibrium player a must subscribe less than $\hat{C}-C_a$ immediately. But then the game shifts to a position where $C_a<\hat{C}$, $C_b<\hat{C}$ and it is b's turn to move, and by the logic just given, b contributes no more than $\hat{C}$ in total. This means that the project is not completed, since $2\hat{C}<K$.[11] The above implies that player a must subscribe precisely $\hat{C}$ in total in any subgame-perfect equilibrium continuation. It follows that a must subscribe $\hat{C}-C_a$ immediately, since if he does then b will complete the project immediately in his turn. (The usual equilibrium argument, similar to the one employed in the proof of Proposition 3.1, shows that although b is indifferent between completing the project immediately and contributing just enough so that the state becomes $(\hat{C},\hat{C})$, in equilibrium b must complete the project.)

We have established that the path described in (ii) of the proposition is the only candidate for an equilibrium path. It is easy to see that it can be supported as an equilibrium path. Finally, it is straightforward to show that in the case $\hat{C}=0$, i.e. $K=V(1-\delta)$ there are two equilibrium paths. In one equilibrium path player 1 completes the project in his turn, while in the other he subscribes zero and player 2 completes the project. ||

Acknowledgement. We thank Ed Lazear, John Moore, Zvi Safra, Bob Wilson, the referees, and especially David Kreps and Ariel Rubinstein for very helpful comments. Support from the following sources is gratefully acknowledged: Batterymarch Financial Management, Boston University, the National Science Foundation (grant SES-8909520), the Sloan Foundation, the Recanati School of Business Administration at Tel-Aviv University, the Stanford Program in Finance, and the University of Western Ontario.

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11. In fact, this implies that there is no subgame-perfect equilibrium the project uncompleted with positive probability. If there were such an equilibrium, then from some point along the path of the equilibrium there would be probability arbitrarily close to one that the project is not completed, and then one can show that a deviation by either player to give enough so that the other player completes the project must give a higher payoff.