

# Efficient Mechanism Design\*

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## Abstract

We study Bayesian mechanism design in situations where agents' information may be multi-dimensional, concentrating on mechanisms that lead to efficient allocations. Our main result is that a generalization of the well-known Vickrey-Clarke-Groves mechanism maximizes the planner's "revenue" among all efficient mechanisms. This result is then used to study multiple object auctions in situations where bidders have privately known "demand curves" and extended to include situations with complementarities across objects or externalities across bidders. We also illustrate how the main result may be used to analyze the possibility of allocating both private and public goods efficiently when budget balance considerations are important. The generalized VCG mechanism, therefore, serves to unify many results in mechanism design theory.

## 1 Introduction

Mechanism design theory now forms an integral part of modern economics.<sup>1</sup> Its varied applications include the theory of monopoly pricing, optimal tax theory and the provision of public goods. Perhaps its most successful application has been to the theory of auctions, a central theme of this paper. With a few exceptions, however, mechanism design theory in general, and auction theory in particular, to date has

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<sup>1</sup>For instance, see the extensive and unified account of mechanism design theory in Mas-Colell *et al.* (1995). This is also an excellent source for references.

restricted attention to the case when agents' private information is one dimensional. Consideration of multi-dimensional types poses substantial technical difficulties and results of any generality seem to be non-existent. The complications are not of a technical nature alone: many of the economic insights from the single-dimensional theory do not extend in any natural fashion. (See Armstrong (1996), McAfee and McMillan (1988) and Wilson (1993)).

An important strand of traditional mechanism design theory is concerned with the design of *optimal* mechanisms (or auctions), that is, mechanisms/auctions which maximize the expected revenue of the seller (Myerson (1981), Wilson (1993), among others). A second strand of the theory, however, is concerned with the design of *efficient* mechanisms (Vickrey (1961), Mirrlees (1971), Groves (1973), among others). Here the primary objective of the planner is not revenue maximization but rather social efficiency.

Our approach combines the two considerations, placing greater weight on efficiency. We are interested in revenue maximizing mechanisms but restrict attention to mechanisms that allocate efficiently. Our main result derives the revenue maximizing mechanism (or auction) in the class of efficient mechanisms (auctions): this is a generalization of the well-known Vickrey-Clarke-Groves (VCG) mechanism. Thus while optimal auctions are difficult to characterize when types are multi-dimensional, we are able to make progress in this area by focusing on efficient mechanisms and are able to extend the extant theory to multi-dimensional types.

With this characterization in hand, we can then address a variety of other questions. For instance, we show that budget balancing mechanisms exist *if and only if* the generalized VCG mechanism runs an expected surplus for the planner.

The fact that the generalized VCG provides a necessary and sufficient condition for the existence of balanced budget efficient mechanism results in simple and unified proofs of some impossibility results concerning (a) efficient bilateral trade; and (b) efficient provision of public goods.

Our set up is abstract. The abstraction leads to a simple formulation and also has the virtue that most of our results are then derived in a very general setting. For example, in the context of multiple object auctions we make very weak assumptions on the structure of the valuations of the bidders. In particular, our results are applicable in situations of non-identical goods with complementarities, issues that have been the focus of much recent attention in auction theory.

The analysis of multiple object auctions forms both a major motivation for this paper and a major area of application of the results.

**Multiple Object Auctions and Privatization:** Although Vickrey's (1961) classic paper on auctions was concerned with the question of efficiency, starting with Myerson (1981), auction theory has primarily been occupied with the question of revenue. Attempts to extend Myerson's beautiful results to the case of multiple objects have been largely unsuccessful. Indeed, in order make the problem tractable,

most of the research in this area has made the assumption that even with multiple objects the private information of the buyers is still one dimensional (Wilson (1979), Maskin and Riley (1990), Ausubel and Cramton (1995), Branco (1996) among others).

The recent spurt in interest in the study of multiple object auctions is largely the result of large scale privatization of socially held assets by governments. These include the sales of industrial enterprises in Eastern Europe and the former Soviet Union, radio spectra in the United States and the railway system in Britain. There is a vast literature in this area. Recent work on the theoretical aspects of privatization includes Maskin (1992), McMillan (1994) and Milgrom (1997).

The primary goal of these privatizations is not the maximization of revenue but rather that the assets being sold are allocated efficiently. Of course, because of the vast sums involved the revenue question is not without interest but it does seem to be of secondary import.

Why should efficiency be the primary goal of privatization? It may be argued that governments should not be concerned about efficiency at all since there is likely to be secondary market for these assets and that transactions in these secondary markets will in the end result in an efficient allocation. Thus, following this line of argument, governments may as well maximize revenue and not worry about how the assets are allocated.

However, because of the nature and size of the assets being privatized, secondary trades are likely to be the result of bilateral (or possibly multilateral) bargaining rather than from trade on well functioning thick markets. And this bargaining will take place under conditions of incomplete information. But it is well understood that bargaining under incomplete information is unlikely to lead to efficient trade (as shown by Myerson and Satterthwaite (1983), for instance). Thus, it may be argued that in order to ensure that the assets are allocated efficiently, the government cannot rely on secondary markets.

Indeed, in the well publicized sale of radio spectra in the United States, the Federal Communications Commission was specifically mandated by Congress to sell the licenses in a way that ensured an “efficient and intensive use of the electromagnetic spectrum.” The designers of the auction rules interpreted this as “putting licenses in the hands of those who value them the most,” (Milgrom (1997)). Revenue issues, while not entirely absent, played a subsidiary role at best.

It is well-known that the twin goals of optimality (revenue maximization) and social efficiency (welfare maximization) may be in conflict. For instance, consider the design of an optimal auction for a single good as in Myerson (1981). First, the optimal auction allocates the object to the agent with the highest “priority level” (see Myerson (1981)) and since this is not necessarily the person who assigns the highest value to the object, the allocation may be inefficient. Second, even in cases where the ranking according to priority levels agrees with the ranking according to values, the optimal auction typically involves a reserve price. Now with positive probability

the seller retains the object even though there are unrealized gains from trade. Thus the optimal auction need not be socially efficient.

**The Main Results:** Our paper derives the following results in the context of independent multi-dimensional types and quasi-linear preferences:

1. *Revenue Maximization:* Our main result is that a generalized version of the well-known Vickrey-Clarke-Groves mechanism maximizes the planners' expected revenue among all efficient mechanisms (Theorem 1 below) that are incentive compatible and individually rational. This result is derived in a very general context.
2. *Multiple Object Auctions:* We show how the main result may be applied to multiple object auctions. In situations where the objects are identical and bidders have decreasing marginal valuations ("downward sloping demand curves") it implies that the Vickrey (1961) auction maximizes the seller's revenue among all auctions that are efficient (Proposition 1). We then show how our abstract set up can also accommodate, very simply, non-identical objects, complementarities ("synergies") among the objects and consumption externalities.
3. *Budget Balance:* Even though the VCG mechanism typically does not balance the planner's budget, we show that there exists an efficient, incentive compatible and individually rational mechanism that also balances the budget if and only if the generalized VCG mechanism runs an expected surplus (Theorem 2). The proof is constructive.
4. *Other Applications:* The result on necessary and sufficient conditions for budget balance is applied to two allocation problems, one with private goods and one with public goods. It leads to very elementary proofs of (a) the impossibility of an efficient trade of a single object among a buyer and a seller (Proposition 2); and (b) the impossibility of efficient provision and financing of a public project (Proposition 3).

Our analysis depends crucially on a technical result which shows that any two incentive compatible mechanisms which implement the same allocation rule must be "payoff equivalent," that is, the expected payoff to an agent can differ in the two mechanisms by at most an additive constant (Lemma 1 below). This result is thus a generalization of Myerson's (1981) payoff equivalence result to the case of when agents' private information is multi-dimensional. It is derived without making any differentiability assumptions on the mechanism and is thus applicable to situations (e.g., auctions) where such assumptions are not natural. The proof of Lemma 1 is relegated to an appendix.

## 2 Preliminaries

There is a set  $K$  of a finite number of social alternatives. Suppose that  $K = \{1, 2, \dots, K\}$  with generic element  $k$ . There are  $I$  individual agents, indexed  $i = 1, 2, \dots, I$ .

Each agent  $i$  then has a  $K$ -dimensional *type*  $t_i = (t_i(1), t_i(2), \dots, t_i(K)) \in \mathbf{R}^K$ . The payoff to agent  $i$  of type  $t_i$  is *quasi-linear*, that is, it is of the form

$$t_i(k) - x_i$$

where  $x_i$  is a monetary transfer made by  $i$  to a planner or a central agency. Let  $T_i$  denote the set of possible types for  $i$ . We assume that for all  $i$ ,  $T_i$  is a compact and convex subset of  $\mathbf{R}^K$ .

For example, in a problem of how to allocate a finite number of indivisible objects, the alternatives may represent allocations of the objects to the individuals and the type of an agent may represent how much value or utility the agent derives from the various alternatives. Notice that the abstract model is general enough to allow for a variety of specifications including complementarities among goods and externalities among agents.

As usual, the vector  $t = (t_1, t_2, \dots, t_I)$  denotes the types of all agents and the vector  $t_{-i} = (t_1, t_2, \dots, t_{i-1}, t_{i+1}, \dots, t_I)$  the types of all agents other than  $i$ . Correspondingly,  $T = \times_{j=1}^I T_j$  and  $T_{-i} = \times_{j \neq i} T_j$  denote the sets of all types and all types other than  $i$ , respectively. The vector  $(s_i, t_{-i}) = (t_1, t_2, \dots, t_{i-1}, s_i, t_{i+1}, \dots, t_I)$ .

Let  $f_i$  denote the density of  $t_i$  on  $T_i$ . We suppose that  $f_i$  is continuous and that the support of  $f_i$ , denoted by  $\text{supp } f_i$ , is  $T_i$ . The types  $t_i$  are assumed to be independently distributed across agents.

The environment considered here is thus one of independent “private values.”

**Mechanisms:** A *direct mechanism* is a pair  $(\kappa, \mu)$  where  $\kappa : T \rightarrow K$  is an *allocation rule* and  $\mu : T \rightarrow \mathbf{R}^I$  is a *payment rule*. Thus, given reports  $s \in T$ ,  $\kappa(s)$  is the chosen alternative and  $\mu_i(s)$  is the transfer payment made by  $i$ . A mechanism  $(\kappa, \mu)$  is said to be (Bayesian) *incentive compatible* if the identity function (“truth-telling”) is a Bayesian equilibrium of the resulting game. By the revelation principle, the restriction to direct mechanisms will be without loss of generality.

Let  $\Delta(K)$  denote the set of probability distributions over  $K$ . Given an allocation rule  $\kappa : T \rightarrow K$ , define  $Q_i(\kappa, \cdot) : T_i \rightarrow \Delta(K)$  as the function whose  $k$ th component is

$$Q_i^k(\kappa, t_i) = \text{Prob} \{s_{-i} \in T_{-i} : \kappa(t_i, s_{-i}) = k\}, \quad (1)$$

that is, the *conditional probability* that alternative  $k$  will be chosen by  $\kappa(\cdot)$  when agent  $i$ 's type is  $t_i$ .<sup>2</sup>

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<sup>2</sup>Although we have assumed that the allocation rule  $\kappa : T \rightarrow K$  is deterministic it would make no difference in what follows if it were random, that is,  $\kappa : T \rightarrow \Delta(K)$ . In that case,  $Q_i^k(t_i)$  would again be defined as before but its computation would incorporate the randomness of the allocation rule also.

In much of what follows, we will fix an allocation rule  $\kappa(\cdot)$  and thus, in order to economize on notation, suppress the dependence of  $Q_i$  on  $\kappa(\cdot)$  by writing  $Q_i(t_i)$  instead of  $Q_i(\kappa, t_i)$ .

Given a mechanism, the *expected payoff* to agent  $i$  from reporting  $s_i$  when his type is  $t_i$  and all other agents are reporting truthfully is

$$E_{t_{-i}}[t_i(\kappa(s_i, t_{-i})) - \mu_i(s_i, t_{-i})] = Q_i(s_i) \cdot t_i - m_i(s_i) \quad (2)$$

where

$$m_i(s_i) = E_{t_{-i}}[\mu_i(s_i, t_{-i})]$$

is the *expected payment* of  $i$  when reporting  $s_i$ .

It is convenient to define the *equilibrium payoff function*

$$U_i(\kappa, \mu, t_i) = Q_i(t_i) \cdot t_i - m_i(t_i). \quad (3)$$

It may be useful to think of  $U_i$  as an indirect utility function.

Once again, in order to economize on notation, we suppress the dependence of  $U_i$  on the specific mechanism  $(\kappa, \mu)$  and write  $U_i(t_i)$  instead of  $U_i(\kappa, \mu, t_i)$ .

Our first result, Lemma 1 below, provides a characterization of the equilibrium payoff functions  $U_i$  that can result from an incentive compatible mechanism. It shows that the equilibrium payoff functions from two such mechanisms with the same allocation rule  $\kappa(\cdot)$  can differ at most by an additive constant. It is a generalization of the “revenue equivalence” result (Myerson (1981)) and Riley and Samuelson (1981) to the case of multi-dimensional types. Under stronger hypotheses, the conclusion of Lemma 1 has been obtained by for general allocation rules by Armstrong (1996) and Jehiel et al. (1996), and for the particular case of efficient rules, by d’Aspremont and Gérard-Varet (1979b) and Williams (1997)

**Lemma 1** (Payoff Equivalence) *Suppose the mechanism  $(\kappa, \mu)$  is incentive compatible. Then the expected payoff function  $U_i$  is determined by  $Q_i$  up to an additive constant. For all  $t_i, s_i \in T_i$ ,*

$$U_i(t_i) = U_i(s_i) + \int_0^1 Q_i(rt_i + (1-r)s_i) \cdot (t_i - s_i) dr. \quad (4)$$

**Proof.** See the Appendix. ■

Since  $U_i(t_i) = Q_i(t_i) \cdot t_i - m_i(t_i)$ , we have that  $U_i(0) = -m_i(0)$ . Then (4) can be rewritten as,

$$m_i(t_i) = m_i(0) + Q_i(t_i) \cdot t_i - \int_0^1 Q_i(rt_i) \cdot t_i dr, \quad (5)$$

which just states that all incentive compatible mechanisms with the same allocation rule  $\kappa(\cdot)$  are also “revenue (or payment) equivalent” up to an additive constant.

Until now our analysis has been fairly standard and followed conventional mechanism design theory. In what follows our concern shifts to *efficient* mechanisms.

### 3 Efficient Mechanisms

We now examine properties of incentive compatible mechanisms such that the allocation rule is socially efficient.

Since agents' payoffs are quasi-linear, an allocation rule is socially efficient if and only if the chosen alternative maximizes the sum of agents' payoffs.

**Definition 1** *An allocation rule  $\kappa^* : T \rightarrow K$  is (ex post) efficient if for all  $t \in T$ , the chosen alternative  $\kappa^*(t)$  maximizes social welfare  $\sum_{i=1}^I t_i(k)$  over  $K$ .<sup>3</sup>*

We call any mechanism  $(\kappa^*, \mu)$  with an efficient allocation rule  $\kappa^*(\cdot)$  an *efficient mechanism*.

It is convenient to define

$$SW(t) = \sum_{i=1}^I t_i(\kappa^*(t)) \quad (6)$$

as the maximized value of *social welfare* from an efficient allocation when the types are  $t$ , and

$$SW_{-i}(t) = \sum_{j \neq i} t_j(\kappa^*(t)) \quad (7)$$

as the *social welfare* of individuals other than  $i$  from an efficient allocation when the types are  $t$ .

### 4 Participation Constraints

Thus far we have concentrated on incentive compatibility, implicitly assuming that all agents participate in the workings of the mechanism, that is, provide their private information,  $t$ , and pay  $\mu(t)$ . In many applications, however, it is natural to assume that agents have outside options and that if their expected payoff from the mechanism is not superior to their outside option, agents can choose to not participate in the workings of the mechanism. Exercising their option to not participate in the mechanism may violate social efficiency.

For instance, consider the problem of allocating a single indivisible good by means of an auction. If there is an individual  $i$  whose interim expected payoff from participating is worse than his outside option he will not participate in the auction. Now, ex post, there will be instances where  $i$  has the highest valuation and the socially efficient allocation would be to award  $i$  the object. However, this fact cannot be determined since his private information,  $t_i$ , is not available to the planner.

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<sup>3</sup>There may be more than one alternative that maximizes social welfare. Since such ties occur with zero probability, which one is chosen will not affect our results.

Thus, in such situations an *efficient* mechanism must guarantee that individuals will not be worse off by participating in the mechanism. This constrains the range of feasible mechanisms.

In many situations it is simplest to model this by assuming that every agent's payoff from not participating is 0. This may be appropriate, for instance, when considering the auction of a privately consumed good with no consumption externalities. In other situations, however, it is more appropriate to model the payoff from not participating as itself being *type dependent* (and hence private information). For instance, in an allocation problem where one of the agents is a producer of some goods the decision to participate may depend on the cost of production itself, that is, on the "type" of the agent.

We will suppose that the "break-even" expected payoff of agent  $i$  is exogenously given by a *continuous* function  $IR_i : T_i \rightarrow \mathbf{R}$ .

**Definition 2** A mechanism  $(\kappa, \mu)$  is (interim) individually rational if for all  $i$  and  $t_i \in T_i$ ,

$$U_i(t_i) \geq IR_i(t_i).$$

Our specification is general enough to allow for the possibility that  $IR_i$  also depends on the allocation rule  $\kappa^*$ . This may be the result of externalities exerted by participants on *non*-participants. As an example, suppose that a valuable technology is being allocated to one of the firms in an industry by means of an auction. Since how this technology is allocated will affect competition among all firms in the industry, including those that choose not to participate in the auction, the underlying allocation rule may affect the expected payoff from not participating.

An extensive treatment of auctions with externalities and a discussion of participation constraints in such an environment can be found in Jehiel et. al (1996).

## 5 The VCG Mechanism and a Generalization

In this section we study the properties of a well-known mechanism due to Vickrey (1961) and Clarke (1971). We then provide a generalization.

### 5.1 The VCG Mechanism

In his pioneering study of auctions Vickrey (1961) introduced a mechanism to efficiently allocate  $L$  identical multiple objects in a context where each bidder has declining marginal values for the objects (or in other words, a downward sloping demand function). In the Vickrey auction each of  $I \geq 2$  bidders submits  $L$  bids in decreasing order and the  $l$ th bid represents the marginal amount the bidder is willing to pay for the  $l$ th object. Out of the  $I \times L$  bids the  $L$  highest bids are awarded objects

and of course,  $I \times (L - 1)$  bids are rejected. If bidder  $i$  gets  $l_i$  objects, he is asked to pay the  $l_i$  highest rejected bids that are not his own. Vickrey (1961) showed that in his auction it was a weakly dominant strategy to bid one's vector of marginal values honestly and as a result the auction was efficient.

Subsequently, in his analysis of the efficient provision of public goods, Clarke (1971) introduced the so called "pivotal mechanism" that leads to an efficient allocation in that context. Clarke (1971) showed that in the pivotal mechanism it was a dominant strategy to report one's value of the public good honestly and the mechanism was efficient.

These mechanisms were later generalized by Groves (1973). The Vickrey (1961) auction and the Clark (1971) pivotal mechanism are, in fact, just special cases of a single abstract mechanism that we will refer to as the Vickrey-Clarke-Groves (or VCG) mechanism.

The VCG mechanism, denoted by  $(\kappa^*, \mu^V)$ , is defined by the payment rule:

$$\begin{aligned} \mu_i^V(t) &= \sum_{j \neq i} t_j(\kappa^*(0, t_{-i})) - \sum_{j \neq i} t_j(\kappa^*(t)) \\ &= SW_{-i}(0, t_{-i}) - SW_{-i}(t) \end{aligned} \tag{8}$$

where  $\kappa^*(0, t_{-i})$  is an efficient alternative that would result if  $i$  were to report  $t_i = 0$  (or equivalently, in many settings, if  $i$  were not present).

Observe that the amount  $\mu_i^V(t)$  represents the *externality* that  $i$  exerts on the other  $I - 1$  agents by his presence in society. It is the difference between the welfare of the others "without him" and the welfare of the others "with him." Notice that in both the Vickrey (1961) auction for private goods and the Clarke (1971) mechanism for public goods each agent pays the externality he exerts on the other  $I - 1$  agents.

Fix some  $t_{-i}$ , the types of agents other than  $i$ . In the VCG mechanism the payoff to  $i$  of type  $t_i$  when he reports  $s_i$  is:

$$\sum_{j=1}^I t_j(\kappa^*(s_i, t_{-i})) - \sum_{j \neq i} t_j(\kappa^*(0, t_{-i})) \tag{9}$$

The second term is independent of the report  $s_i$  and the first is maximized by choosing  $s_i = t_i$ . Thus, as is well known, "truth-telling" is a weakly dominant strategy in the VCG mechanism. Thus, *a fortiori*, the VCG mechanism is *incentive compatible*.

Using (9) agent  $i$ 's *ex post* payoff in equilibrium (when  $s_i = t_i$ ) is just  $SW(t) - SW(0, t_{-i})$ , that is, the difference in social welfare when  $i$  reports  $t_i$  versus when he reports 0.

## 5.2 The Generalized VCG Mechanism

The payment in the VCG mechanism is exactly the externality that  $i$  exerts on other agents from reporting  $t_i$  rather than zero (see (8)). In this subsection we use this

interpretation to generalize the original VCG mechanism. This generalization will play a central role in our analysis.

Fix a vector of types  $s = (s_1, s_2, \dots, s_n) \in T$ , one for each player. The *VCG mechanism with basis  $s$* , is defined by

$$\begin{aligned} \mu_i^*(t \mid s_i) &= s_i(\kappa^*(s_i, t_{-i})) + \sum_{j \neq i} t_j(\kappa^*(s_i, t_{-i})) - \sum_{j \neq i} t_j(\kappa^*(t)) - IR_i(s_i) \\ &= [SW(s_i, t_{-i}) - SW_{-i}(t)] - IR_i(s_i) \end{aligned} \quad (10)$$

It is routine to verify that truth-telling is a weakly dominant strategy in the generalized VCG mechanism and thus it is also incentive compatible.

Agent  $i$ 's *ex post* payoff in equilibrium is

$$[SW(t) - SW(s_i, t_{-i})] + IR_i(s_i)$$

which is just the *difference* in social welfare that would result if  $i$  were to report  $t_i$  rather than  $s_i$  plus the individually rational level of type  $s_i$ .

The corresponding expected payoff is

$$U_i^*(t_i \mid s_i) = E_{t_{-i}}[SW(t) - SW(s_i, t_{-i})] + IR_i(s_i) \quad (11)$$

Observe that if for all  $i$ , we have  $s_i = 0$  and  $IR_i(s_i) = 0$ , then the VCG mechanism with basis  $s_i$  defined in (10) is the same as the original VCG mechanism in (8).

### 5.3 Optimal Choice of Basic Types

We have defined the VCG mechanism relative to any basis  $s$ . In what follows it will be important to choose  $s$  optimally.

Define

$$\begin{aligned} \underline{t}_i &\in \arg \min_{t_i \in T_i} E_{t_{-i}} \left[ \sum_{j=1}^I t_j(\kappa^*(t)) \right] - IR_i(t_i) \\ &= \arg \min_{t_i \in T_i} E_{t_{-i}} [SW(t)] - IR_i(t_i) \end{aligned} \quad (12)$$

and consider the mechanism  $\mu^*(t \mid \underline{t}_i)$ , that is, the *VCG mechanism with basis  $\underline{t}$* . The type  $\underline{t}_i$  is the “most reluctant” type of agent  $i$  in the sense that his gain from participating in any VCG mechanism is the least among all types of agent  $i$ .

Using (11) we can write:

$$\begin{aligned} U_i^*(t_i \mid \underline{t}_i) &= Q_i^*(t_i) \cdot t_i - m_i^*(t \mid \underline{t}_i) \\ &= Q_i^*(t_i) \cdot t_i + E_{t_{-i}} [SW_{-i}(t) - SW(\underline{t}_i, t_{-i})] + IR_i(\underline{t}_i) \\ &= E_{t_{-i}} [SW(t) - SW(\underline{t}_i, t_{-i})] + IR_i(\underline{t}_i) \end{aligned}$$

so that

$$U_i^* (\underline{t}_i | \underline{t}_i) = IR_i (\underline{t}_i).$$

Next, using (12), we can write, for all  $t_i$ ,

$$E_{t_{-i}} [SW (t)] - IR_i (t_i) \geq E_{t_{-i}} [SW (\underline{t}_i, t_{-t})] - IR_i (\underline{t}_i) \quad (13)$$

and by rearranging (13) we obtain that

$$\begin{aligned} U_i^* (t_i | \underline{t}_i) &= E_{t_{-i}} [SW (t) - SW (\underline{t}_i, t_{-t})] + IR_i (\underline{t}_i) \\ &\geq IR_i (t_i) \end{aligned}$$

so that the mechanism  $\mu^* (\cdot | \underline{t})$  is individually rational.

To summarize, we have argued that for a VCG mechanism with basis  $\underline{t} = (\underline{t}_1, \underline{t}_2, \dots, \underline{t}_n)$  defined in (12) we have that for all  $t_i$ ,

$$\begin{aligned} U_i^* (t_i | \underline{t}_i) &\geq IR_i (t_i) \\ U_i^* (\underline{t}_i | \underline{t}_i) &= IR_i (\underline{t}_i). \end{aligned} \quad (14)$$

## 5.4 Payment Maximizing Efficient Mechanisms

Our main result is:

**Theorem 1** (Payment Maximization) *Among all mechanisms that are efficient, incentive compatible and individually rational, the VCG mechanism with basis  $\underline{t}$ , defined in (12) maximizes the expected payments of each agent.*

**Proof.** As above, let  $U_i^* (\cdot | \underline{t}_i)$  denote the expected payoff function for the VCG mechanism with basis  $\underline{t}$ ,  $(\kappa^*, \mu^* (\cdot | \underline{t}))$ .

Now suppose  $(\kappa^*, \mu)$  is any other efficient mechanism that is incentive compatible and individually rational. If  $U_i (\cdot)$  is the expected payoff function corresponding to  $(\kappa^*, \mu)$  then by the payoff equivalence result (Lemma 1), for all  $t_i$ ,

$$U_i (t_i) - U_i^* (t_i | \underline{t}_i) = U_i (\underline{t}_i) - U_i^* (\underline{t}_i | \underline{t}_i).$$

Individual rationality requires that for all  $i$ ,  $U_i (\underline{t}_i) \geq IR_i (\underline{t}_i) = U_i^* (\underline{t}_i | \underline{t}_i)$  and consequently, for all  $t_i$ ,

$$U_i (t_i) \geq U_i^* (t_i | \underline{t}_i).$$

Thus, for all  $t_i$ ,

$$m_i (t_i) \leq m_i^* (t_i | \underline{t}_i).$$

Thus the VCG mechanism with basis  $\underline{t}$  maximizes the expected payment of the agents. ■

It is worthwhile to note that in the class of efficient Bayesian mechanisms the revenue maximizing mechanism has a dominant strategy equilibrium.<sup>4</sup> In general, in order to compute the optimal basis,  $\underline{t}$ , as in (12), the planner needs to know the distribution of types  $f_i$ . In many natural applications, however, knowledge of  $f_i$  may be unnecessary. For example, if, as often assumed in auction theory,  $T_i = \mathbf{R}_+^K$  and  $IR_i(t_i) = 0$  then regardless of the density  $f_i$ ,  $\underline{t}_i = 0$ .

## 6 Multiple Object Auctions

In this section we illustrate how the main result of the previous section (Theorem 1) may be applied to study auctions of multiple objects in the original environment studied by Vickrey (1961).

Suppose that there are  $K$  identical objects to be auctioned and each bidder has a downward sloping “demand curve” which is privately known. Let  $t_i(k) \geq 0$  denote the *marginal value* of the  $k$ th object assigned by bidder  $i$  of type  $t_i$ . Thus if  $i$  were to obtain  $L$  objects the total value would be  $\sum_{k=1}^L t_i(k)$ . It is natural to call the vector  $t_i$  the “demand curve” of bidder  $i$ . The set of types  $T_i$  is then just the set of possible downward sloping demand curves so that  $T_i = \{t_i \in \mathbf{R}_+^K : t_i(1) \geq t_i(2) \geq \dots \geq t_i(K)\}$ . Suppose further that each bidder’s payoff from not participating in the auction is 0.

Clearly, in this environment the types  $\underline{t}_i = 0$ , and thus the VCG mechanism with basis  $\underline{t}$  is exactly the auction proposed by Vickrey (1961).<sup>5</sup>

The following result on multiple object auctions is an immediate consequence of Theorem 1:

**Proposition 1** (Revenue Maximization) *Suppose all bidders have downward sloping demand curves. The Vickrey auction maximizes the expected revenue of the seller among all auctions that are efficient, incentive compatible and individually rational.*

While Proposition 1 applies to the environment originally studied by Vickrey (1961), Theorem 1 is applicable to a more general class of auction problems. For instance, in the case of identical objects it applies not only when individual demand curves are downward sloping but even when they are not. Moreover, Theorem 1 applies as well to situations when the objects are not necessarily identical and even

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<sup>4</sup>Mookherjee and Reichelstein (1992) study the possibility of using dominant strategy mechanisms in place of Bayesian mechanisms in a general environment. In particular, they are not concerned with efficient rules alone.

<sup>5</sup>Recently, Ausubel (1995) has proposed an open ascending bid auction which, in this environment, is outcome equivalent to a Vickrey auction.

allows for the possibility that there are complementarities (or synergies) in consumption. Indeed, it is general enough to accommodate externalities in consumption across bidders. This is because in our abstract set up a particular social alternative,  $k$ , can denote a complete allocation of the objects to bidders and a particular bidder may then be sensitive to how the objects are allocated to his rivals. Finally, we have allowed for the possibility that the participation constraints are type dependent.

## 7 Budget Balance

In many economic problems a *desideratum* of a mechanism is that for every realization of types, the transfers from agents sum to zero, that is, the planner's budget is exactly balanced *ex post*. In our notation, a mechanism is said to balance the budget if for all  $t$ ,

$$\sum_{i=1}^I \mu_i(t) = 0.$$

We know from the work of Green and Laffont (1977) that no dominant strategy mechanism can always balance the budget. However, Arrow (1979) and d'Aspremont and Gérard-Varet (1979a) independently showed that there do exist Bayesian incentive compatible mechanisms with the balanced budget property.

The Arrow (1979) and d'Aspremont and Gérard-Varet (1979a) (or AGV) mechanism (also called the “expected externality” mechanism)  $(\kappa^*, \mu^A)$  is defined by

$$\begin{aligned} \mu_i^A(t) &= \left(\frac{1}{I-1}\right) \sum_{j \neq i} E_{s_{-j}} [SW_{-j}(t_j, s_{-j})] - E_{s_{-i}} [SW_{-i}(t_i, s_{-i})] \\ &= \left(\frac{1}{I-1}\right) \sum_{j \neq i} E_{s_{-j}} \left[ \sum_{l \neq j} s_l(\kappa^*(t_j, s_{-j})) \right] - E_{s_{-i}} \left[ \sum_{j \neq i} s_j(\kappa^*(t_i, s_{-i})) \right] \end{aligned} \quad (15)$$

so that for all  $t$ ,  $\sum_{i=1}^I \mu_i^A(t) = 0$ .

To see that the AGV mechanism is incentive compatible note that the expected payoff to  $i$  from reporting  $s_i$  when his type is  $t_i$  when all other agents are reporting truthfully is:

$$E_{t_{-i}} \left[ t_i(\kappa^*(s_i, t_{-i})) + \sum_{j \neq i} t_j(\kappa^*(s_i, t_{-i})) \right] - \left(\frac{1}{I-1}\right) E_{t_{-i}} \left[ \sum_{j \neq i} E_{s_{-j}} [SW_{-j}(t_j, s_{-j})] \right]$$

and since the second term is independent of  $s_i$ , this is maximized by setting  $s_i = t_i$ .

It is easy to see that the AGV mechanism may not satisfy the individual rationality constraint. The question of whether there are efficient, incentive compatible, individually rational mechanisms which also balance the budget can also be answered by means of the VCG mechanism.

**Theorem 2** *There exists an efficient, incentive compatible and individually rational mechanism that balances the budget if and only if the VCG mechanism with basis  $\underline{t}$  results in an expected surplus, that is, if and only if*

$$E_t \left[ \sum_{i=1}^I \mu_i^* (t \mid \underline{t}) \right] \geq 0. \quad (16)$$

**Proof.** The fact that (16) is necessary follows from Theorem 1 above: if the VCG mechanism with basis  $\underline{t}$  runs a deficit then all efficient, incentive compatible and individually rational mechanisms must run a deficit.

We now show that (16) is sufficient by explicitly constructing a mechanism that balances the budget and is individually rational. First, consider the AGV mechanism  $\mu^A$  defined in (15). From Lemma 1 we know that there exist constants  $c_i^A$  such that:

$$U_i^A (t_i) = E_{t_{-i}} [SW (t)] - c_i^A$$

Now consider the VCG mechanism with basis  $\underline{t}$ ,

$$\mu_i^* (t \mid \underline{t}) = SW (t_i, t_{-i}) - SW_{-i} (t) - IR_i (t_i)$$

Again, from Lemma 1 we know that there exist constants  $c_i^*$  such that

$$U_i^* (t_i \mid \underline{t}_i) = E_{t_{-i}} [SW (t)] - c_i^*$$

Next, suppose the VCG mechanism with basis  $\underline{t}$  runs an expected surplus, that is,

$$E_t \left[ \sum_{i=1}^I \mu_i^* (t \mid \underline{t}) \right] \geq 0$$

Then we have that

$$E_t \left[ \sum_{i=1}^I \mu_i^* (t \mid \underline{t}) \right] \geq E_t \left[ \sum_{i=1}^I \mu_i^A (t) \right]$$

since the right hand side is exactly 0.

Equivalently,

$$\sum_{i=1}^I c_i^* \geq \sum_{i=1}^I c_i^A \quad (17)$$

For all  $i > 1$ , define  $d_i = c_i^A - c_i^*$  and let  $d_1 = -\sum_{i=2}^I d_i$ . Consider the mechanism  $\bar{\mu}$  defined by

$$\bar{\mu}_i (t) = \mu_i^A (t) + d_i$$

Clearly,  $\bar{\mu}$  balances the budget. It is also incentive compatible since the payoff to each agent in the mechanism  $\bar{\mu}$  differs from the payoff from an incentive compatible mechanism,  $\mu^A$ , by a constant. It remains to verify that  $\bar{\mu}$  is individually rational.

For all  $i \neq 1$

$$\begin{aligned}\bar{U}_i(t_i) &= U_i^A(t_i) + d_i \\ &= U_i^A(t_i) + c_i^A - c_i^* \\ &= U_i^*(t_i) \\ &\geq IR_i(t_i)\end{aligned}$$

By construction  $\sum_{i=1}^I d_i = 0$  and observe from (17) that

$$d_1 = -\sum_{i=2}^I d_i = \sum_{i=2}^I (c_i^* - c_i^A) \geq (c_1^A - c_1^*)$$

and thus

$$\begin{aligned}\bar{U}_1(t_1) &= U_1^A(t_1) + d_1 \\ &\geq U_1^A(t_1) + c_1^A - c_1^* \\ &= U_1^*(t_1) \\ &\geq IR_1(t_1)\end{aligned}$$

so that  $\bar{\mu}$  is also individually rational. ■

**Corollary 1** *If for all  $i$ ,  $IR_i \equiv 0$  and all  $t_i \geq 0$  then there exists an efficient, incentive compatible and individually rational mechanism that balances the budget.*

Under these assumptions, for each  $i$ ,  $\underline{t}_i = 0$  and thus the VCG mechanism with basis 0 is

$$\begin{aligned}\mu_i^*(t \mid 0) &= SW(0, t_{-i}) - SW_{-i}(t) \\ &\geq 0\end{aligned}$$

since, by definition,  $SW(0, t_{-i}) = \max_k \sum_{j \neq i} t_j(k)$ .

Thus the VCG mechanism with basis 0 always runs a surplus and the result follows from Theorem 2.

Consider the problem of allocating a bundle of *private* goods among a group of agents in a way that ensures budget balance. For instance, a bundle of scarce resources may need to be efficiently allocated among the divisions of a company. While an auction will accomplish this task there may be situations where a balanced budget mechanism is more natural. Corollary 1 implies that such a mechanism always exists. Indeed, such a mechanism is constructed explicitly in the proof of Theorem 2.

## 8 Inefficiency Results

The revenue maximization result (Theorem 1 above) shows that the expected payments in the VCG mechanism  $m_i^*(t_i)$  are an upper bound to the expected payments from any efficient and individually rational mechanism. In this section we illustrate how this result may be applied in two settings that are very different from that of an auction.

### 8.1 Bilateral Trade

First, we show that the inefficiency result of Myerson and Satterthwaite (1983) follows rather simply as a consequence of the revenue maximization result.

Consider a situation of trade between a seller (agent 1) and a buyer (agent 2). There is a single indivisible good owned by the seller. There is also a divisible good, money, and as usual, payoffs are quasi-linear. Let the set of alternatives be  $K = \{1, 2\}$  where  $k = 1$  denotes that there is no trade and the seller keeps the good whereas  $k = 2$  denotes that there is trade and the buyer gets the object.

The seller's set of types is  $T_1 = [0, 1] \times \{0\}$  and the buyer's set of types is  $T_2 = \{0\} \times [0, 1]$ . A seller of type  $t_1$  will not participate unless he gets at least his payoff from no trade and thus  $IR_1(t_1) = t_1(1)$ . For the buyer  $IR_2(t_2) = 0$  for all  $t_2 \in T_2$ .

Clearly, from the definition of efficiency  $\kappa^*(t) = 1$  if  $t_1(1) > t_2(2)$  and  $\kappa^*(t) = 2$  if  $t_1(1) < t_2(2)$ . Thus,  $SW(t) = t_1(1)$  if  $t_1(1) > t_2(2)$  and  $SW(t) = t_2(2)$  if  $t_1(1) < t_2(2)$ .

Now clearly from (12), we have that  $\underline{t}_1 = (1, 0)$  and  $\underline{t}_2 = (0, 0)$ .

A *trading mechanism* is a mechanism  $(\kappa, \mu)$  where for each  $t \in T$ ,  $\kappa(t)$  is an allocation and  $\mu_i(t)$  are payments satisfying  $\mu_1(t) + \mu_2(t) = 0$ . Thus a trading mechanism is a mechanism in which there is no net inflow or outflow of funds and the planner's budget is balanced.

As in Myerson and Satterthwaite (1983) we ask whether there exists a trading mechanism that is individually rational and promotes efficient trade among the agents. Denote by  $\kappa^*(t)$  the allocation that is efficient when the types are  $t$ .

We know that the VCG mechanism with basis  $\underline{t}$ ,  $(\kappa^*, \mu^*(\cdot | \underline{t}))$  is efficient. Let us now compute the payments by the two agents in the VCG mechanism with basis  $\underline{t}$ . It is easy to see that if  $t_1(1) > t_2(2)$  then

$$\begin{aligned}\mu_1^*(t | \underline{t}) &= 0 \\ \mu_2^*(t | \underline{t}) &= 0\end{aligned}$$

whereas if  $t_1(1) < t_2(2)$  then

$$\begin{aligned}\mu_1^*(t | \underline{t}) &= -t_2(2) \\ \mu_2^*(t | \underline{t}) &= t_1(1)\end{aligned}$$

Thus

$$\begin{aligned}\mu_1^*(t) + \mu_2^*(t) &= -t_2(2) + t_1(1) \\ &\leq 0\end{aligned}$$

and is strictly less than zero unless the no-trade is itself efficient. Thus the VCG mechanism with basis  $\underline{t}$  runs an expected deficit. From Theorem 2, we infer that *no* individually rational mechanism will promote efficient trade unless the mechanism injects money into the economy, that is, it cannot be a trading mechanism.<sup>6</sup> Thus we obtain:

**Proposition 2** *In the bilateral trading problem there does not exist an efficient, incentive compatible and individually rational trading mechanism.*

Williams (1997) also provides an alternative derivation of the Myerson and Satterthwaite result which is closely related. In particular, he shows an equivalence between any efficient mechanism and one in the Groves class; and that no Groves mechanism can satisfy individual rationality and balance the budget simultaneously. He then uses an equivalence result to analyze both the possibility and impossibility of efficient trade, especially in situations that involve many agents. McAfee (1991) and Makowski and Mezzetti (1994) also use similar techniques.

These papers, however, do not identify the salience of the VCG mechanism.

## 8.2 Public Project Choice

Next, consider the classic public goods problem. Suppose that a public good can be either provided or not. Thus the set of social alternatives is  $K = \{0, 1\}$ . Let  $t_i \geq 0$  denote agent  $i$ 's utility from the public good so that for each individual,  $T_i = [0, 1]$ . Suppose that for all  $i$  and  $t_i$ ,  $IR_i(t_i) = 0$  so that  $\underline{t}_i = 0$ .

Suppose that it costs  $c$  to produce the public good. It is efficient to undertake the project ( $k = 1$ ) if and only if  $\sum_{j=1}^I t_j \geq c$ .

It is known that in no dominant strategy mechanism (such as a VCG mechanism) can the cost of the public good exactly equal the payments of the agents. In other words, in such mechanisms the budget cannot be balanced. The AGV mechanism proposed by Arrow (1979) and d'Aspremont and Gérard-Varet (1979a) satisfies the budget balancing property but need not be individually rational. Mailath and Postlewaite (1991) show that no mechanism that balances the budget can be individually rational.<sup>7</sup>

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<sup>6</sup>For the sake of exposition we have assumed that the supports of the values of the seller and the buyer are the same. It is routine to confirm that the result continues to hold as long as the supports overlap (as in Myerson and Satterthwaite (1983)).

<sup>7</sup>Laffont and Maskin (1979) also allude to the difficulty of achieving *interim* individual rationality and demonstrate the possibility of achieving *ex ante* individual rationality, that is,  $E[U_i(t_i)] \geq 0$ .

To see how this last result is also a consequence of Theorem 2 let us, once again, compute the payments in the VCG mechanism. We have that the externality exerted by  $i$  on the other  $I - 1$  agents is, in the usual notation:

$$\mu_i^*(t) = \left[ \sum_{j \neq i} t_j(\kappa^*(t_{-i})) - c\kappa^*(t_{-i}) \right] - \left[ \sum_{j \neq i} t_j(\kappa^*(t)) - c\kappa^*(t) \right]$$

and thus  $\mu_i^*(t) > 0$  only if  $\kappa^*(t_{-i}) = 0$  and  $\kappa^*(t) = 1$ , that is, only if  $i$  is “pivotal.” Now, when  $\mu_i^*(t) > 0$ ,

$$\mu_i^*(t) = \left[ c - \sum_{j \neq i} t_j \right] \leq t_i$$

because  $k = 1$  only if  $\sum_{j=1}^I t_j \geq c$ . If with positive probability, it is strictly better to provide the good than not, then the expected payment in VCG mechanism is less than  $c$ . Since the VCG mechanism maximizes expected payments among all efficient and individually rational mechanisms we obtain:

**Proposition 3** *Suppose that with positive probability, it is strictly better to provide the public good than not. Then there does not exist an efficient and individually rational mechanism which always covers the cost of producing the public good.*

## 9 Conclusion

We have shown how a single mechanism, the VCG mechanism with an optimally chosen basis, provides a unified perspective on many areas of mechanism design theory. It raises the greatest revenue among auctions that result in an efficient allocation of multiple objects. It also serves to determine the possibility of achieving efficiency when other considerations (e.g. budget balance) are present.

## 10 Appendix

In this appendix we derive some consequences of incentive compatibility. Our goal is to provide a proof for the payoff equivalence result (Lemma 1).

**Definition 3** *The direct mechanism  $(\kappa, \mu)$  is incentive compatible if for all  $i$  and for all  $t_i, s_i \in T_i$ :*

$$Q_i(t_i) \cdot t_i - m_i(t_i) \geq Q_i(s_i) \cdot t_i - m_i(s_i). \quad (18)$$

First, observe that incentive compatibility is equivalent to the statement that for all  $t_i$  and  $s_i$  in  $\text{supp } f_i$  (which is assumed to be  $T_i$ ):

$$\begin{aligned} U_i(t_i) &\geq Q_i(s_i) \cdot t_i - m_i(s_i) \\ &= Q_i(s_i) \cdot s_i - m_i(s_i) + Q_i(s_i) \cdot (t_i - s_i) \\ &= U_i(s_i) + Q_i(s_i) \cdot (t_i - s_i). \end{aligned} \tag{19}$$

By interchanging the roles of  $t_i$  and  $s_i$  in (19) we also have

$$U_i(s_i) \geq U_i(t_i) + Q_i(t_i) \cdot (s_i - t_i), \tag{20}$$

and together (19) and (20) yield:

$$Q_i(t_i) \cdot (s_i - t_i) \leq U_i(s_i) - U_i(t_i) \leq Q_i(s_i) \cdot (s_i - t_i). \tag{21}$$

Second, observe that incentive compatibility can be rewritten as:

$$U_i(t_i) = \max_{s_i} \{Q_i(s_i) \cdot t_i - m_i(s_i)\}$$

For each  $s_i$ , the expression in the brackets is an affine function of  $t_i$  and since the maximum of a family of affine functions is a convex function,  $U_i$  is convex (Rochet (1987)). Since  $Q_i$  is bounded, taking limits in (21) as  $s_i \rightarrow t_i$  establishes that  $U_i$  is continuous everywhere. (The fact that  $U_i$  is convex already implies that it is continuous on the relative interior of its domain.) Thus incentive compatibility implies that  $U_i$  is *convex and continuous*.

**Proof of Lemma 1.** Suppose  $(\kappa, \mu)$  is incentive compatible. In order to establish (4) fix a  $t_i$  and a  $s_i$  in  $T_i$  and define a function  $V_i : [0, 1] \rightarrow \mathbf{R}$  by

$$V_i(r) = U_i(rt_i + (1-r)s_i)$$

so that  $V_i(0) = U_i(s_i)$  and  $V_i(1) = U_i(t_i)$ . Since  $U_i : T_i \rightarrow \mathbf{R}$  is convex and continuous and  $V_i : [0, 1] \rightarrow \mathbf{R}$  is also convex and continuous.

A convex function is absolutely continuous (Lemma 5.16 in Royden (1968)) and thus it is differentiable almost everywhere (with respect to Lebesgue measure) in the interior of its domain. Furthermore, every absolutely continuous function is the integral of its derivative (Theorem 5.13 in Royden (1968)) and so we have:

$$V_i(1) = V_i(0) + \int_0^1 V_i'(r) dr. \tag{22}$$

Now suppose  $r \in (0, 1)$  is such that  $V_i$  is differentiable at  $r$ . From (20):

$$\begin{aligned} V_i(r + \delta) - V_i(r) &= U_i((r + \delta)t_i + (1 - r - \delta)s_i) - U_i(rt_i + (1 - r)s_i) \\ &\geq Q_i(rt_i + (1 - r)s_i) \cdot \delta(t_i - s_i). \end{aligned}$$

If  $\delta > 0$  then we get:

$$\frac{V_i(r + \delta) - V_i(r)}{\delta} \geq Q_i(rt_i + (1 - r)s_i) \cdot (t_i - s_i)$$

and taking the limit as  $\delta \downarrow 0$  we obtain that  $V_i'(r) \geq Q_i(rt_i + (1 - r)s_i) \cdot (t_i - s_i)$ . On the other hand, if  $\delta < 0$  we get the opposite inequality, and taking the limit as  $\delta \uparrow 0$  we obtain that  $V_i'(r) \leq Q_i(rt_i + (1 - r)s_i) \cdot (t_i - s_i)$ . Thus if  $V_i$  is differentiable at  $r \in (0, 1)$ ,

$$V_i'(r) = Q_i(rt_i + (1 - r)s_i) \cdot (t_i - s_i).$$

Now substituting in (22) we obtain that for all  $t_i \in T_i$ :

$$U_i(t_i) = U_i(t'_i) + \int_0^1 Q_i(rt_i + (1 - r)t'_i) \cdot (t_i - t'_i) dr$$

Thus at any point in  $T_i$ ,  $U_i$  is determined by  $Q_i$  up to an additive constant. This completes the proof. ■

If  $U_i$  were differentiable everywhere then from (21) we would have  $\nabla U_i = Q_i$ . In that case  $U_i$  would be a *potential* of the vector field  $Q_i$  and because the line integral of a function with a potential is path independent (Apostol (1969)) we could write:

$$U_i(t_i) = U_i(0) + \int Q_i \cdot d\alpha$$

for any piecewise smooth path  $\alpha : [0, 1] \rightarrow T_i$  such that  $\alpha(0) = 0$  and  $\alpha(1) = t_i$ . The conclusion of Lemma 1 would then be immediate.

But  $U_i$  need not be differentiable as the following simple example shows.

**Example 1:** Suppose that there are only two agents ( $I = 2$ ) and  $K = \{1, 2, 3\}$ . Let  $T_1 = [0, 1] \times \{0\} \times \{0\}$  and  $T_2 = \{0\} \times [0, 2] \times [0, 2]$  and let the types be uniformly distributed on  $T_1$  and  $T_2$ . Consider an efficient allocation rule  $\kappa^*$ , that is, for all  $t$ ,  $\kappa^*(t) \in \arg \max t_1(k) + t_2(k)$  and suppose that the payment functions are  $\mu_i(t) = -t_j(\kappa^*(t))$  where  $j \neq i$ . The mechanism  $(\kappa^*, \mu)$  is incentive compatible (truth-telling is a dominant strategy) but if  $\min \{t_2(2), t_2(3)\} > 1$ ,  $U_2(t_2) = \max \{t_2(2), t_2(3)\}$  and thus  $U_2$  is not differentiable at any  $t_2$  such that  $t_2(2) = t_2(3) > 1$ .

Also note that in the example above, the resulting  $Q_2$  is discontinuous at any  $t_2$  such that  $t_2(2) = t_2(3) > 1$ .

We emphasize that Lemma 1, is derived without making any assumptions about the differentiability of  $U_i$  and thus applies even to the example given above.

Finally, we point out that the assumption that the set of types  $T_i$  is convex (and that  $\text{supp } f_i = T_i$ ) plays an important role in Lemma 1.

**Example 2:** Suppose that there is an auction of a single object to two agents. Let the types indicate the value that each agent assigns to the object and suppose that  $T_1 = [0, 3]$  whereas  $T_2 = [0, 1] \cup [2, 3]$ . Let the types be uniformly distributed over  $T_1$  and  $T_2$ . First, consider the VCG mechanism  $(\kappa^*, \mu^*)$  (which in this environment is the same as a second-price auction). It may be verified that the associated expected payoff function for agent 2 is:

$$U_2^*(t_2) = \frac{1}{6}t_2^2.$$

Second, consider a mechanism  $(\kappa^*, \mu)$  which is the same as a second-price auction except that if agent 2 reports a type  $t_2 \in [2, 3]$ , he pays an amount  $\frac{1}{6}$  (as an “entry fee”) regardless of whether he wins the auction or not. The mechanism  $(\kappa^*, \mu)$  is also incentive compatible and individually rational (but note that truth-telling is not a dominant strategy). The associated expected payoff function is:

$$U_2(t_2) = \begin{cases} \frac{1}{6}t_2^2 & \text{if } t_2 \in [0, 1] \\ \frac{1}{6}t_2^2 - \frac{1}{6} & \text{if } t_2 \in [2, 3] \end{cases}$$

Now  $U_2^*$  and  $U_2$  differ by more than an additive constant. Moreover, the expected payments in  $(\kappa^*, \mu)$  exceed the expected payments in the VCG mechanism  $(\kappa^*, \mu^*)$ .

Thus the conclusions of Lemma 1 need not obtain if the sets  $T_i$  are not convex.

**Related work** As mentioned above, the conclusion of Lemma 1 is well known as the “revenue equivalence theorem” and was derived under very weak hypotheses by Myerson (1981) in an environment where types are single dimensional. (Remarkably, an early example of revenue equivalence can be found in Vickery’s (1961) classic paper.)

Note that Lemma 1 is valid for *any* allocation rule  $\kappa$ , not just for efficient allocation rules  $\kappa^*$ . For efficient rules, the result has been derived in a multi-dimensional setting by d’Aspremont and Gérard-Varet (1979b) under stronger hypotheses that involve the differentiability of the mechanism  $(\kappa^*, \mu)$ . These hypotheses preclude the application of their results to auctions.

Williams (1997) presents a generalization of d’Aspremont and Gérard-Varet’s (1979b) equivalence result for efficient rules. In particular, he does not require the mechanism to be differentiable. Instead, he directly assumes that the payoff function  $Q_i(s_i) \cdot t_i - m_i(s_i)$  is differentiable in  $s_i$  at  $s_i = t_i$ . Example 2 above shows that even this rather weak assumption need not be satisfied.<sup>8</sup>

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<sup>8</sup>There are other technical differences between Williams’ (1997) framework and ours. His is more geared toward trading problems rather than auctions. We view his work and ours as complementary.

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