UTILITARIAN PRICING OF ANNUITIES

by

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Abstract

In a perfectly competitive market for annuities with full information , the prices of annuities are equal to individuals' (discounted) survival probabilities (*'separating equilibrium'*). That is, prices are actuarially fair for each individual. In contrast, the pricing implicit in social security systems invariably allows for cross subsidization between different risk classes (e.g. males/females). We apply the theory of optimum commodity taxation to examine the utilitarian approach to the pricing of annuities and show how the solution depends on the joint distribution of survival probabilities and incomes in the population. Specifically, a low correlation between survival probabilities and incomes leads, under utilitarianism, to subsidization (taxation) of individuals with high (low) survival probabilities. We also consider whether it is desirable to affect the uniform equilibrium price of a *'pooling equilibrium'* (individual characteristics being private information). In this case, the policy instrument is an income tax whose proceeds are used to subsidize the purchase of annuities. As before, the desirability of annuity subsidization depends on the covariance between incomes and survival probabilities.

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I Introduction: First-Best Allocation

In a perfectly competitive market for annuities with full information (*'Separating Equilibrium'*), the prices of annuities are equal to individuals' (discounted) survival probabilities. That is, prices are *actuarially fair.* In contrast, the pricing implicit in social security systems invariably allows for cross subsidization between different risk classes, implying transfers from high to low risk individuals. For example, most social security systems provide the same benefits to males and females of equal age and with equal income and retirement histories inspite of the higher life expectancy of females[1].

We want to examine the utilitarian approach to this issue using the theory of optimum commodity taxation. Consider a population that consists of H individuals. Denote the expected utility of individual h by U_h, $h = 1, 2, ..., H$. Utilitarianism attempts to maximize a social welfare function, W, which depends on the U_h's:

$$W = W\ (U_1,\ U_2, ...,\ U_H). \tag{1}$$

W depends positively on, and is assumed to be differentiable, symmetric and concave in, the U_h's.

Each individual lives for either one or two periods, and individuals differ in their survival probabilities. Let p_h be the probability that individual h lives for two periods; c_{1h} be consumption of individual h in period 1 and c_{2h} be consumption of individual h in period 2, if he or she is then alive. Utility derived from consumption, $c > 0$, by any individual in any period during life is $u(c)(> 0)$. It is the same in either period so there is no *time preference.* When not alive, utility is 0. Expected utility of individual h is thus

$$U_h = u(c_{1h}) + p_h u(c_{2h}). \tag{2}$$

The economy has a given amount of resources, R, which can be used in either period and they can be carried forwards without any gain or loss. With a large number of individuals, expected consumption in the two periods must therefore equal the given resources:

$$(3) \qquad \sum_{h=1}^{H} c_{1h} + \sum_{h=1}^{H} p_h c_{2h} = R$$

Maximization of (1) s.t. (3) yields the condition that consumption is equal in both periods, $c_{1h} = c_{2h} = c_h$, for all $h = 1, 2, ..., H$. Consequently, expected utility, (2), becomes $U_h = (1 + p_h)u(c_h)$ and the resource constraint, (3), becomes

$$(4) \qquad \sum_{h=1}^{H} (1 + p_h)c_h = R.$$

The *First-Best* optimum allocation of consumption, c_h, among individuals is obtained by maximizing the welfare function (1) subject to the resource constraint, (4). The F.O.C. are

$$(5) \qquad W_h u'(c_h) = \text{ constant, } \textit{for all } h = 1, 2, ..., H.$$

where $W_h = \dfrac{\partial W}{\partial U_h}$. Denote the solutions to (4)-(5) by $c_h^*(\underline{p})$, $\underline{p}= (p_1, p_2, .., p_H)$, the corresponding optimum expected utilities by $U_h^* = (1+p_h)u(c_h^*)$ and $W^* = W(U_1^*, U_2^*, ..U_n^*,)$.

It can be shown that for any $j, k = 1, 2, ..H$, $U_j^* \gtreqless U_k^*$ as $p_j \gtreqless p_k$. To demonstrate this, take $H = 2$. Write the resource constraint (4) in terms of (U_1, U_2) :

$$(6) \qquad (1 + p_1)v(\frac{U_1}{1 + p_1}) + (1 + p_2)v(\frac{U_2}{1 + p_2}) = R$$

where the function v is implicitly defined by $U_h = (1+p_h)u(v)$. Hence, $v' > 0$ and $v'' < 0$. The implicit relation between U_1 and U_2 defined by (6) is strictly convex and it's absolute slope is equal to $v'(\frac{U_1}{1+p_1})/v'(\frac{U_2}{1+p_2})$. Hence, along the $U_1 = U_2$ line this slope is $\gtreqless 1$ as $p_1 \lesseqgtr p_2$ (Figure 1). Symmetry of W implies that the slope of social indifference curves, $W_0 = W(U_1, U_2)$, along the 45^0 line is unity, and hence $U_1^* \gtreqless U_2^* \Longleftrightarrow p_1 \gtreqless p_2$.

The ranking of optimum consumption levels, $c_h^*(\underline{p})$, depends on more specific properties of the welfare and utility functions. For example, for an additive social welfare function, $W = \sum\limits_{h=1}^{H} U_h$, (1)-(5) imply that $c_h^* = \frac{R}{\sum\limits_{h=1}^{H}(1+p_h)}$, while

$$U_h^* = (1+p_h)\ u\left(\frac{R}{\sum\limits_{h=1}^{H}(1+p_h)}\right).$$

Thus, the utilitarian *First-Best* has inequality in expected utilities, but may have equality in consumption levels [Arrow, 1992].

This result is similar to Mirrlees' optimum income tax model [1971] where individuals differ in their productivity[2]. The *First-Best* allocation provides higher (expected) utility to those with higher capacity to produce utility.

It is shown in appendix A that

$$0 > \frac{1+p_n}{c_h^*}\frac{\partial c_h^*}{\partial p_h} > -1 \tag{7}$$

while $\frac{\partial c_h^*}{\partial p_j} \gtreqless 0$, for $j \neq h, h, j = 1, 2, .., n$.

Concavity of u and (7) imply

$$\frac{\partial U_h^*}{\partial p_h} = u(c_h^*) + (1+p_n)u'(c_n)\frac{\partial c_h^*}{\partial p_n} > 0 \tag{8}$$

while $\dfrac{\partial U_h^*}{\partial p_j} \gtreqless 0,\, j \neq h,\ j = 1, 2, .., H.$[3] Thus, with given total resources, an increase in one individual's survival probability decreases his/her optimum consumption, but the positive effect of higher survival probability on expected utility dominates. The effect on the welfare of other individuals facing only resource redistribution depends on the shape of the social welfare function.

II Competitive Annuity Market with Full Information

In a competitive market with full information on individuals' survival probabilities and a zero rate of interest, the price of a unit of second period consumption, c_{2h}, is equal to the survival probability of each annuitant. Individuals maximize expected utility subject to a budget constraint

$$c_{1h} + p_h c_{2h} = y_h \qquad h = 1,\ 2, .., \ H \tag{9}$$

where y_h is the given income of individual h. Demands, for first and second period consumption (annuities), $\widehat{c}_{1h}$ and $\widehat{c}_{2h}$, are given by $\widehat{c}_{1h} = \widehat{c}_{2h} = \widehat{c}_h = \dfrac{y_h}{1+p_h}$.

The *First-Best* allocation can be supported by a competitive annuity market accompanied by an optimum income allocation. Equating consumption levels under competition, $\widehat{c}_h$, to the optimum levels, $c_h^*(\underline{p})$, yields *unique* income levels, $\widehat{y}_h = (1 + p_h)\ c_h^*(\underline{p})$, that support the *First-Best* allocation. In particular, with an additive W, $c_h^* = \dfrac{R}{\sum\limits_{h=1}^{H}(1+p_h)}$, hence

$$\widehat{y}_h = \frac{1+p_h}{\sum\limits_{h=1}^{H}(1+p_h)} R \tag{10}$$

III Second Best Optimum Pricing of Annuities

Governments do not engage, for well-known reasons, in unconstrained lump-sum redistributions of incomes. In contrast, most annuities are supplied directly by government-run social security systems and taxes/subsidies can, if so desired, be applied to prices of annuities offered by private pension funds. These prices can be used by governments to improve social welfare. Although deviations from actuarially fair prices entail distortions (i.e. efficiency losses), distributional improvements may outweigh the costs.[4]

Suppose that individual h purchases annuities at a price of q_h. With an income y_h, his or her budget constraint is

$$c_{1h} + q_h c_{2h} = y_h, \qquad h = 1, 2, ..., H. \tag{11}$$

Maximization of (2) subject to (11) yields demands $\widehat{c}_{ih} = \widehat{c}_{ih}(q_h, p_h, y_h)$, $i = 1, 2$, and $h = 1, 2, ..., H$. Maximized expected utility, $\widehat{U}_h$, is $\widehat{U}_h(q_h, p_h, y_h) = u(\widehat{c}_{1h}) + p_h u(\widehat{c}_{2h})$.

Assume that no outside resources are available for the annuity market, hence total subsidies/taxes must equal zero,

$$\sum_{h=1}^{H} (q_h - p_h)\widehat{c}_{2h} = 0 \tag{12}$$

Maximization of $W(\widehat{U}_1, \widehat{U}_2, ..., \widehat{U}_H)$ *w.r.t.* prices $(q_1, ..., q_H)$ subject to (12) yields F.O.C.

$$\frac{\partial W}{\partial \widehat{U}_h} \frac{\partial \widehat{U}_h}{\partial q_h} + \lambda \left[\widehat{c}_{2h} + (q_h - p_h) \frac{\partial \widehat{c}_{2h}}{dq_h} \right] = 0, \qquad h = 1, 2, ..., H, \tag{13}$$

where $\lambda > 0$ is the shadow price of constraint (12). In elasticity form, using Roy's identity $\left(\frac{\partial \widehat{U}_h}{\partial q_h} = -\frac{\partial \widehat{U}_h}{\partial y_h} \widehat{c}_{2h} \right)$, (13) can be written

$$\frac{q_h - p_h}{q_h} = \frac{\theta_h}{\varepsilon_h} \tag{14}$$

where $\varepsilon_h = -\frac{q_h}{\widehat{c}_{2h}} \frac{\partial \widehat{c}_{2h}}{dq_h}$ is the *price elasticity* of second period consumption of individual h, and $\theta_h = 1 - \frac{1}{\lambda} \frac{\partial W}{\partial \widehat{U}_h} \frac{\partial \widehat{U}_h}{\partial y_h}$ is the *net* social value of a marginal transfer to individual h through the optimum pricing scheme. Equation (14) is a variant of the well-known *inverse-elasticity* optimum tax formula, which combines equity (θ_h) and efficiency $(\frac{1}{\varepsilon_h})$ considerations.

The implication of (14) for the optimum pricing of annuities depends on the welfare function, W, and on the joint distribution of incomes, $(y_1, ..., y_H)$, and probabilities, $(p_1, ...p_H)$.

To obtain some concrete examples, let W be the sum of expected utilities. Then $\frac{\partial W}{\partial \widehat{U}_h} = 1$, $h = 1, 2, ..., H$. Assume further that $U_h = \ln c_{1h} + p_h . \ln c_{2h}$ In this case, demands are,

$$\widehat{c}_{1h} = \frac{y_h}{1 + p_h} \quad ; \quad \widehat{c}_{2h} = \frac{y_h}{1 + p_h} \frac{p_h}{q_h} \tag{15}$$

and

$$\widehat{U}_h = (1 + p_h) \ln \left(\frac{y_h}{1 + p_h} \right) + p_h \ln \left(\frac{p_h}{q_h} \right) \tag{16}$$

Conditions (14) and (12) now yield the solution

$$q_h = \phi \left(\frac{\beta_h}{\sum\limits_{h=1}^{H} \beta_h} \right), \tag{17}$$

where $\phi = \sum_{h=1}^{H} p_h > 0$ and $\beta_h = \dfrac{p_h y_h}{1+p_h} > 0$.

Consider two special cases of (17):

(a) Equal Incomes: $(y_h = y = \dfrac{R}{H}; \quad h = 1, 2, .., H)$.

Condition (17) now becomes $q_h = \bar{\phi}\left(\dfrac{p_h}{1+p_h}\right)$, where $\bar{\phi} = \dfrac{\sum_{h=1}^{H} p_h}{\sum_{h=1}^{H}\left(\dfrac{p_h}{1+p_h}\right)} (> 1)$.

It is seen (*Figure 1*) that optimum pricing involves subsidization (taxation) of individuals with high (low) survival probabilities.[5]

(b) $y_h = y(1+p_h)$

This, one recalls, is the First-Best utilitarian income distribution and since all price elasticities are equal to unity, we see from (17), as expected, that $q_h = p_h$, i.e., efficiency prices are optimal.

More generally, it is seen from (17) that a higher correlation between incomes, y_h, and survival probabilities, p_h, decreases - and possibly eliminates - the subsidization of high survival individuals. In contrast, a negative correlation between incomes and survival probabilities (as, presumably, in the female/male case) leads to subsidies for high survival individuals, possibly to the commonly observed uniform pricing rule.

IV Competitive Annuity Market: Pooling Equilibrium with Subsidization

Information on individual survival probabilities is typically unavailable or only imperfectly inferred by the issuers of annuities. In these circumstances, the only practical case (and possibly also the optimal case if grouping into *'risk-classes'* involves significant errors) is to charge all individuals a common price, q, for annuities. The equilibrium price is determined by a zero expected profits condition. This is called a *'Pooling Equilibrium'*.

Demands are now denoted by $\tilde{c}_{ih}(q,\, p_h,\, y_h)$, $i = 1,\, 2$ and the zero expected profits condition is $\sum\limits_{h=1}^{H}(q - p_h)\tilde{c}_{2h}(q,\, p_h,\, y_h) = 0$, or

$$(18) \qquad q = \frac{\sum p_h \tilde{c}_{2h}(q,\, p_h,\, y_h)}{\sum \tilde{c}_{2h}(q,\, p_h,\, y_h)} = \sum \alpha_h\, p_h$$

where $\alpha_h = \dfrac{\tilde{c}_{2h}}{\sum \tilde{c}_{2h}} \geq 0$, $\sum\limits_{h=1}^{H} \alpha_h = 1$. The price of annuities in a pooling equilibrium is a weighted average of the population's survival probabilities, the weights being the (equilibrium) demands for annuities. Equation (18) is an implicit equation to determine the equilibrium q as a function of $\underline{p}$ and $\underline{y}.$ Denote this solution by $\tilde{q}$. Under standard conditions, $\tilde{q}$ is unique and stable (Appendix B).

Since $\dfrac{\partial \tilde{c}_{2h}}{\partial q} < 0$, clearly all individuals for which $\tilde{q} < p_h$ $(> p_h)$ purchase a larger (smaller) amount of annuities than in the full-information market equilibrium. This is the *'adverse-selection'* which characterizes a pooling equilibrium.

Denote optimum utilities in a pooling equilibrium by $\tilde{U}_h = u(\tilde{c}_{1h}) + p_h u(\tilde{c}_{2h})$. A pooling equilibrium raises the utilities of those with high survival probabilities and decreases utilities of those with low survival probabilities compared to the levels of utility in a full-information equilibrium: $\tilde{U}_h > (<)\hat{U}_h$ for high (low) p_h's. The effect on social welfare depends, in particular, on the correlation generated by the joint distribution of incomes and survival probabilities. A negative correlation tends to raise social welfare because those who gain are the 'poor' with high social weights while the wealthy losers have low weights. It is not possible to sign the difference in the levels of social welfare for all possible distributions of survival probabilities and incomes. We shall demonstrate this for some particular cases.

Assume that social welfare is the sum of expected utilities, and that utility of consumption is logarithmic, $u = \ln c$. In a full-information equilibrium $\hat{c}_{1h} = \hat{c}_{2h} = \dfrac{y_h}{1+p_h}$, and $\hat{U}_h = (1+p_h)\ln(\dfrac{y_h}{1+p_h})$. In a pooling equilibrium, $\tilde{c}_{1h} = \dfrac{y_h}{1+p_h}$, $\tilde{c}_{2h} = \dfrac{p_h\ y_h}{(1+p_h)\,q}$ and $\tilde{U}_h = (1+p_h)\ln(\dfrac{y_h}{1+p_h}) + p_h \ln\dfrac{p_h}{q}$.

The weights α_h in the equilibrium formula for $\tilde{q}$, (18), are now $\alpha_h = \dfrac{\dfrac{p_h y_h}{1+p_h}}{\sum \dfrac{p_h y_h}{1+p_h}}$.

When social welfare is $W = \sum\limits_{h=1}^{H} U_h$, we can calculate:

$$\tilde{W} - \hat{W} = \sum_{h=1}^{H} p_h \ln(\frac{p_h}{q}) = \ln\left(\frac{p_1^{p_1} p_2^{p_2} \ldots p_H^{p_H}}{\sum \alpha_h p_h}\right). \tag{19}$$

Since (19) is homogeneous of degree 0 in the p_h's, we normalize $\sum p_h = 1$. At the First-Best income distribution, $y_h = \dfrac{(1+p_h)\,R}{\sum(1+p_h)}$, (19) should be negative. Indeed, in this case (19) is equal to $\ln\left(\dfrac{p_1^{p_1} p_2^{p_2} \ldots p_H^{p_H}}{\sum p_h^2}\right)$ and since the term in brackets is less than one (the geometric average is less than the arithmetic average), we have $\tilde{W} - \hat{W} < 0$.

It is easy, on the other hand, to find distributions of income and survival probabilities that make (19) positive (take, for example, $H = 2$ and y_2 very small). Empirically, although in some groups (e.g. women) the correlation between longevity and incomes is negative, the overall observed correlation is clearly positive, in which case a pooling equilibrium will be socially inferior to a full-information equilibrium.

Let us turn now to possible government involvement in the pricing of annuities. The only potential policy instrument available to the government in order to affect q is to tax/subsidize the purchase of annuities. Consider a proportional income tax at rate t, $0 \leq t \leq 1$.[6]

Tax proceeds are devoted to subsidize the annuity market. The zero-expected profits constraint is now written $\sum\limits_{h=1}^{H}(p_h - q)\widehat{c}_{2h}(q,\, p_h,\, (1-t)y_h) = t\sum\limits_{h=1}^{H} y_h$. Hence, the equation that determines implicitly the equilibrium price, $\tilde{q}(t)$, is

$$\tilde{q} = \tilde{q}(t) = \frac{\sum\limits_{h=1}^{H} p_h \widehat{c}_{2h}(\tilde{q},\, p_h,\, (1-t)y_h) - t\sum\limits_{h=1}^{H} y_h}{\sum\limits_{h=1}^{H} \widehat{c}_{2h}(\tilde{q},\, p_h,\, (1-t)y_h)} \tag{20}$$

Under the conditions that ensure that (18) has unique solution and normality of c_1 and c_2, it can be shown that $\dfrac{d\tilde{q}}{dt} < 0$ (see Appendix B).

Given a joint distribution of survival probabilities and incomes, is it socially desirable to subsidize annuities? We have seen that in a full-information equilibrium it is desirable to subsidize individuals with high survival probabilities when incomes are negatively correlated with survival probabilities. In fact, such subsidization is desirable when the level of this correlation is less than the (positive) one which supports the First-Best. Let us now examine this question in the context of a pooling equilibrium, an additive welfare function and logarithmic utility. For this utility, $\tilde{c}_{1h} = \dfrac{(1-t)y_h}{1+p_h}$, $\tilde{c}_{2h} = \dfrac{(1-t)p_h\, y_h}{(1+p_h)\, q}$ and $\tilde{U}_h = (1+p_h)\ln(\dfrac{y_h}{1+p_h}) + (1+p_h)\ln(1-t) + p_h \ln\dfrac{p_h}{q}$. The equilibrium price, $\tilde{q}$, is now given by

$$\tilde{q} = \frac{\sum \dfrac{p_h y_h}{1+p_h}}{\sum \dfrac{p_h y_h}{1+p_h} + \dfrac{t}{1-t}\sum y_h} \tag{21}$$

All the conditions specified in Appendix B are satisfied and hence a unique equilibrium price exists for any $0 \leq t < 1$.

Social welfare in the pooling equilibrium is $\tilde{W}(t) = \sum\limits_{h=1}^{H} \tilde{U}_h$. Taking (21) into account

$$\left.\frac{d\tilde{W}}{dt}\right|_{t=0} = -\sum_{h=1}^{H}(1+p_h) + \frac{\left(\sum\limits_{h=1}^{H} p_h\right)\left(\sum\limits_{h=1}^{H} y_h\right)}{\sum\limits_{h=1}^{H} \dfrac{p_h y_h}{1+p_h}} \tag{22}$$

Hence, $\left.\dfrac{d\tilde{W}}{dt}\right|_{t=0} > 0$ iff $\sum\limits_{h=1}^{H} \dfrac{p_h}{\sum\limits_{h=1}^{H} p_h} \dfrac{y_h}{\sum\limits_{h=1}^{H} y_h} \Bigg/ \dfrac{1+p_h}{\sum\limits_{h=1}^{H}(1+p_h)} < 1$.

As in the full-information equilibrium, the correlation between (modified) survival probabilities and incomes determines the desirability of taxation devoted to subsidize the purchase of annuities.

Interestingly, when income distribution is First-Best, that is, $y_h = \dfrac{(1+p_h)R}{\sum\limits_{h=1}^{H}(1+p_h)}$, it can be calculated from (22) that $\left.\dfrac{d\tilde{W}}{dt}\right|_{t=0} = 0$. The optimum has no government involvement[7]. To summarize, although the distribution of utilities in a pooling equilibrium is different from that in a full-information equilibrium for any given income distribution, the conclusion is that in both cases intervention by the government is justified once income distribution deviates from the First-Best distribution (which displays a *positive* correlation between survival probabilities and incomes).

Appendix A

Let $H = 2$. The extention to $H > 2$ is immediate. The F.O.C. for maximization of (1) s.t. (3) are

$$
\begin{aligned}
& & W_1(U_1^*, U_2^*)u'(c_1^*) - \lambda &= 0 & \text{(A.1)}\\
&\text{(23)} & W_2(U_1^*, U_2^*)u'(c_2^*) - \lambda &= 0 & \\
&\text{(24)} & R - (1+p_1)c_1^* - (1+p_2)c_2^* &= 0 &
\end{aligned}
$$

where $U_h^* = (1+p_h)u(c_h^*)$, $h = 1, 2.$.

Totally differentiating (A.1) *w.r.t.* p_1 yields:

$$
\begin{aligned}
\frac{(1+p_1)}{c_1^*}\frac{\partial c_1^*}{\partial p_1} &= \frac{1}{\Delta}\{(1+p_1)(1+p_2)[(W_{11}\,u'(c_1^*)^2 - W_{12}\,u'(c_1^*)u'(c_2^*))/ \qquad \text{(A.2)}\\
& /\frac{c_1^* u'(c_1^*)}{u(c_1^*)} - W_{12}\,u'(c_1^*)u'(c_2^*) + W_{22}\,u'(c_2^*)^2] + \\
& + W_2 u''(c_2^*)(1+p_1)\}
\end{aligned}
$$

where (using (A.1)),

$$
\begin{aligned}
\Delta &= -\frac{(1+p_1)(1+p_2)\lambda^2}{W_1^2 W_2^2}[W_{11}W_2^2 - 2W_{12}W_1W_2 + W_{22}W_1^2] - \qquad \text{(A.3)}\\
& - (1+p_1)W_2 u''(c_2^*) - (1+p_2)W_1 u''(c_1^*)
\end{aligned}
$$

Strict quasi-concavity of W implies that $\Delta > 0$.

Since $0 < \dfrac{c_1^* u'(c_1^*)}{u(c_1^*)} < 1$, inserting again (A.1) into (A.2) we obtain

(A.4)
$$0 > \frac{(1+p_1^*)}{c_1^*}\frac{\partial c_1^*}{\partial p_1} > -1$$

as stated in the text.

Differentiating (A.1) *w.r.t.* p_2,

$$\begin{aligned} \frac{(1+p_2)}{c_2^*}\frac{\partial c_1^*}{\partial p_2} &= \frac{1}{\Delta}\{(1+p_1)(1+p_2)[W_{11}\,u'(c_1^*)^2 - W_{22}\,u'(c_2^*)^2(\frac{u(c_2^*)}{c_2^* u'(c_2^*)}) - \\ &\quad -W_{12}\,u'(c_1^*)u(c_2^*) - W_{12}\,u'(c_1^*)u(c_2^*)(\frac{u(c_2^*)}{c_2^* u'(c_2^*)}) - W_1 u''(c_1^*)\} \end{aligned} \tag{A.5}$$

The first term on the R.H.S. is negative, the second positive, hence the sign of $\frac{\partial c_1^*}{\partial p_2}$ cannot be established in general.

Appendix B

Expected profits, denoted by φ, are

$$\varphi(q) = q\sum_{h=1}^{H}\tilde{c}_{2h}(q, p_h, y_h) - \sum_{h=1}^{H}p_h\tilde{c}_{2h}(q, p_h, y_h) \tag{B.1}$$

Equating (18), $\varphi(q) = 0$, is an implicit equation to determine the equilibrium q. Clearly, $\varphi(\underline{q}) < 0$ where $\underline{q} = \min(p_{1,}\, p_2, .., \, p_H)$ and $\varphi(\bar{q}) > 0$, where $\bar{q} = \max(p_{1,}\, p_2, .., \, p_H)$. Hence, there *exists* at least one q for which $\varphi(q) = 0$. Denote one of these equilibrium q's by $\tilde{q}$. It is assumed that

$$\varphi'(\tilde{q}) = \sum_{h=1}^{H}\tilde{c}_{2h}(\tilde{q}, p_h, y_h) + \sum_{h=1}^{H}(\tilde{q} - p_h)\frac{\partial\tilde{c}_{2h}(\tilde{q}, p_h, y_h)}{\partial q} > 0. \tag{B.2}$$

This is a familiar local stability condition: an increase in the price of c_2 at $\tilde{q}$ raises expected profits and hence competition will drive the price down. The opposite effect occurs when q decreases. It is assumed that (B.2) holds at *any* $\tilde{q}$ which satisfies $\varphi(\tilde{q}) = 0$, implying that $\tilde{q}$ is *unique.*

It is assumed that c_{1h} and c_{2h} are normal goods. Hence, $\frac{\partial\tilde{c}_{2h}}{\partial y} > 0$ and $\frac{\partial\tilde{c}_{1h}}{\partial y} = 1 - q\frac{\partial\tilde{c}_{2h}}{\partial y} > 0.$

Differentiating (20) w.r.t. t yields

$$\frac{d\tilde{q}}{dt}_{|q\,=\,\tilde{q}} = \frac{1}{\varphi'(\tilde{q})}\left(\sum_{h=1}^{H}(q - p_h)\frac{\partial\tilde{c}_2}{\partial y}y_h - \sum_{h=1}^{H}y_h\right) \tag{B.3}$$

By the above assumptions, $\frac{d\tilde{q}}{dt} < 0.$

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Notes

[1] Further subsidization is provided when females are allowed to retire earlier. The best introduction to the broad theoretical issues discussed here is Diamond [2003]. The book by Brown *et-al* [2001] provides a good survey and analysis of the US annuity market.

[2] In Mirrlees' model with additive utilities, the First-Best has all individuals with equal consumption and those with higher productivity, having a lower disutility for generating a given income, are assigned to work more and hence have a lower utility.

[3] In the extreme case when $W = Min[U_1, U_2, .., U_H]$, optimum expected utilities, $U_h^* = (1 + p_h)u(c_h^*)$, are equal and hence optimum consumption, c_h^*, strictly decreases with p_h (and increases with p_j, $j \neq h$).

[4] In practice, of course, prices do not vary individually. Rather, individuals with similar survival probabilities are groped into *'risk-classes'*, and annuity prices and taxes/subsidies vary across these classes.

[5] In *Figure 1*, it can be shown that $\dfrac{\bar{\phi}}{2} < 1$.

[6] A proportional income tax is equivalent to a sales tax on consumption and annuities. This is feasible even if individuals' incomes, as well as survival probabilities, are unknown to the government.

[7] If incomes are equal $y_h = \dfrac{R}{H}$, $h = 1, 2...H$, then it can be seen that $\left.\dfrac{d\tilde{W}}{dt}\right|_{t=0} > 0$, i.e. subsidization of annuities is desirable. This is analogous to subsidization (taxation) of individuals with high (low) survival probabilities in a full-information equilibrium.

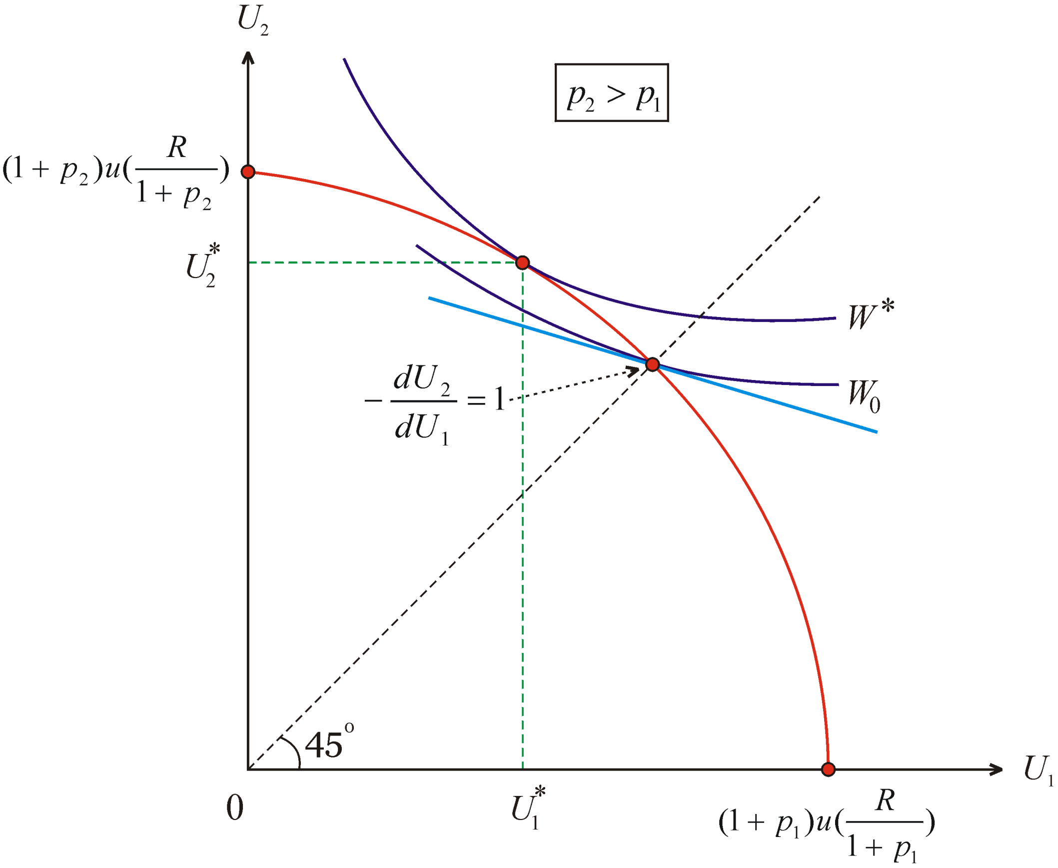
U_2
$p_2 > p_1$
$(1+p_2)u(\frac{R}{1+p_2})$
U_2^*
W^*
W_0
$-\frac{dU_2}{dU_1} = 1$
45°
U_1
0
U_1^*
$(1+p_1)u(\frac{R}{1+p_1})$

Figure I

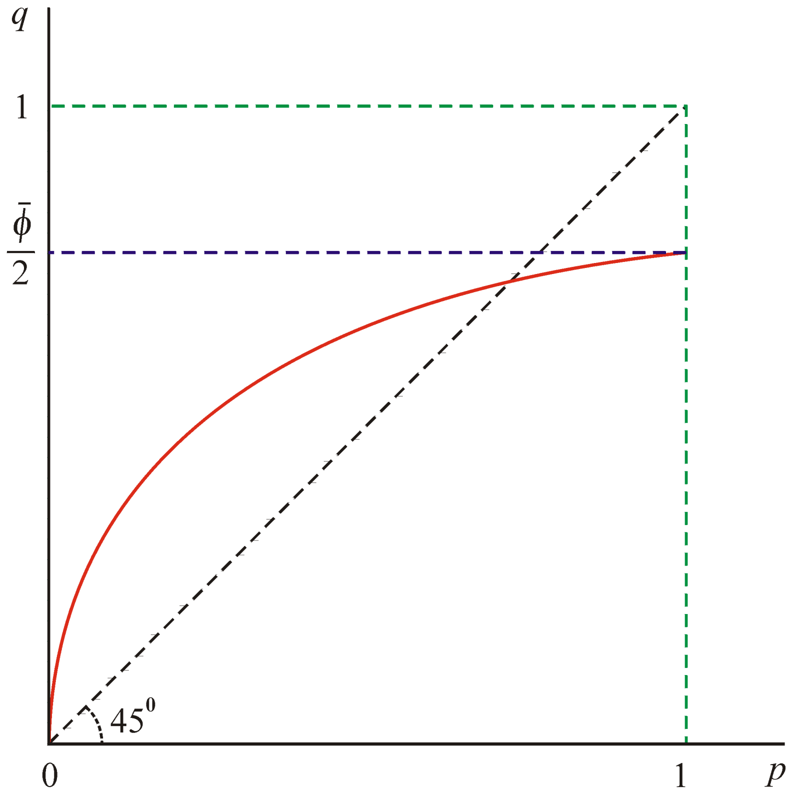

Figure II

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