A Lot of Ambiguity[*]

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Preliminary

Abstract

We consider a risk averse decision maker who dislikes ambiguity as in the Ellsberg urns and compare the certainty equivalent of this gamble with the certainty equivalent of the anchoring probabilistic lottery. We deal first with the Choquet EU model and show that under some conditions on the capacity ν, when independent ambiguous gambles are repeated and the expected value of the anchoring lottery is zero, the difference between the average ambiguous and risky certainty equivalents converges to zero. When the parallel expected value is positive, we show that if the average certainty equivalent of the risky lottery is non-negative, then so is the limit of the average value for the ambiguous model. These results do not extend to the maxmin model or to the smooth recursive model.

Keywords: Ellsberg urns, repeated ambiguity, repeated risk, Choquet expected utility, maxmin

1   Introduction

A patient sees his doctor and it is clear to both of them that a treatment may improve his health. The doctor offers him two possible treatments. A standard, well investigated one, which with probability p leads to a good outcome and with probability $1 - p$ leads to an outcome that is worse than the no-treatment outcome. Alternatively, she offers him a new treatment with somewhat ambiguous probabilities of success. It is however known that whatever the outcome, it improves over that of the standard treatment. Moreover, although the probabilities are not known for sure, they are believed to be somewhere around $p : 1 - p$. The patient is ambiguity averse, and as the improvement in the outcomes of the new treatment is not much, he prefers the old treatment with the known probability of success. In particular, he may prefer the standard treatment to no treatment, which in turn he prefers to the ambiguous one.

The doctor does not have any information she did not share with the patient. Moreover, although she knows that she will see many patients like him, she believes that she won't gain any information about the probability of success of the new treatment, as this probability depends entirely on unobservable charactristics of the patients. Her preferences over risk and uncertain prospects are the same as the patient's (alternatively, she adopts the patient's preferences). Does it follow that she too will prefer the standard treatment to the new one?

Although they have exactly the same information and preferences, there is one dimension in which the patient and the doctor are different, and this is the number of cases they face. The patient sees only one case, his. Ambiguity aversion can be explained as fear of the unknown. Many people believe that they are unlucky and therefore, if they choose the ambiguous prospect, they'll find out that the winning probabilities took a bad turn and are on the lower side of the expectations. But can people really believe that they are always unlucky? The doctor is ambiguity averse, but as she is facing many similar cases, her aversion to each case is probably diminishing. Our aim in this paper is to formalize this intuition. Our main results show that under some assumptions, within the Choquet expected utility model (Schmeidler [26]), the following results hold. If the expected value of the gamble with the known probabilities is zero, then the average certainty equivalents of the repeated ambiguous and the repeated probabilistic gambles converge to each other (Theorem 1). If the expected value of the known gamble is positive, when

2

the certainty equivalents of the probablilstic gamble are non-negative, then in the limit, the average certainty equivalent of the ambiguous lottery becomes non-negative (Theorem 2). And when the utility is bounded from above and the average certainty equivalents of the probabilistic gamble is positive, the ambiguous lotteries eventually become desirable. Then we show that an analogue of Theorem 1 holds within the smooth model, under some other assumptions.

These results lead to immediate policy-making questions. Suppose that the risky and ambiguous treatment have zero expected value. Suppose further that the ambiguous treatment is less expensive than the probabilistic one, but the difference is less than the difference between the two certainty equivalents of the patient. Should society encourage, maybe even force, the use of the ambiguous treatment? Patients may be willing to pay the extra price for the unambiguous treatment, but if we adopt the point of view of care takers (who don't have any better information) we may opt out for the ambiguous treatment. Answers to such questions are beyond the scope of the current paper, but our aim here is to show that they are meaningful and real.

Section 2 presents the structure of our gambles and the Choquet model. The main results are given in section 3. Our results do not extend to some other models, for example, the maxmin expected utility model (Gilboa and Schmeidler [12]) or to the smooth recursive utility model (Klibanoff, Marinacci, and Mukerji [15]). We show this in section 4. We discuss some further issues and the literature in section 6. All claims are proved in the appendix.

2 Setup

An urn contains Γ balls of γ colors. One ball is picked at random, and state of nature s_i is that color i is picked. Denote $S = \{s_1, \ldots, s_\gamma\}$, and define $\Sigma = 2^S$. The number of balls of some colors may be known to be Γ/γ, making the corresponding states of nature probabilistic with probability $\frac{1}{\gamma}$.[1] This ratio also serves as an anchor for non prbabilistic states and events. For example, in the 3-color Ellsberg [3] urn which contains 90 balls, of which 30 are red and each of the other 60 is either black or yellow, the anchoring probabilities

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[1]There are always at least two probabilistic events, $\varnothing$ and S.

are $\frac{1}{3}$ for each of the three colors and $\frac{2}{3}$ for each of complementing events.[2] For more on the anchoring probabilities, see Fox and Tversky [10], Nau [20], Chew and Sagi [2] and Ergin and Gul [6]. For $E = \{s_{i_1}, \ldots, s_{i_\ell}\} \in \Sigma$, let $P(E) = \frac{\ell}{\gamma}$.

Assume now the existence of a sequence of such urns. Let $S_i = S$ be the set of states in urn i with the corresponding algebra $\Sigma_i = \Sigma$. The information regarding each of these urns is the same. Moreover, the outcome or even the mere existence of any urn doesn't change the decision maker's information regarding any other urn. Finally, let $\mathcal{S}^n = S_1 \times \ldots \times S_n$ and $\Omega^n = 2^{\mathcal{S}^n}$. For $E \in \Omega^n$, define $P^n(E)$ to be the number of sequences in E divided by γ^n.

Consider a non-degenerate act $L = (x_1, E_1; \ldots; x_m, E_m)$ where $x_1, \ldots, x_m \in \Re$, $x_1 \leqslant \ldots \leqslant x_m$, $x_1 < x_m$, and $E_1, \ldots, E_m$ is a partition of Σ. Define the anchor lottery $X = (x_1, p_1; \ldots; x_m, p_m)$ where $p_i = P^1(E_i) := P(E_i)$ is the anchor probability of E_i. Denote the expected value of X by μ. The gamble L^n is the sequence of gamble L played once on each of the n urns. We assume that the decision maker is interested in the total sum of outcomes he wins but not in the order or the composition of colors leading to these wins and will therefore view L^n as $(x_1^n, E_1^n; \ldots; x_{k_n}^n, E_{k_n}^n)$, where $x_1^n = nx_1 \leqslant \ldots \leqslant x_{k_n}^n = nx_m$ and E_i^n is the collection of sequences of events from $\Sigma_1, \ldots, \Sigma_n$ such that the sum of their corresponding outcomes is x_i^n. The lottery $X^n = (x_1^n, p_1^n; \ldots; x_{k_n}^n, p_{k_n}^n)$ is a sequence of n independent lotteries of type X where p_i^n is the anchor probability P^n of E_i^n. The lottery X^n serves as a natural anchor for L^n.

Consider now a decision maker having preferences $\succeq^n$ over $\mathcal{L}^n$, the space of all real acts over Ω^n. We assume that the decision maker evaluates lotteries with known probabilities using expected utility theory with the vNM function u. Denote by c^n the certainty equivalent of X^n, that is, the number satisfiying $u(c^n) = \text{EU}(X^n)$. One can view $\text{E}[X^n] - c^n = n\mu - c^n$ as the risk premium the decision maker is willing to pay for trading the lottery X^n for its expected value. Likewise, we define d^n to be the certainty equivalent of L^n, satisfying $d^n \sim^n L^n$ and the ambiguity premium to be the sum the decison maker is willing to pay out of the expected value of the anchor lottery X^n in order to avoid playing L^n, that is, $\text{E}[X^n] - d^n = n\mu - d^n$.

In a famous article, Samuelson [24] showed that the risk premium, or even

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[2]More complicated urns are also possible, for example, an urn containing 100 balls. Twenty of which are yellow, and each of the others is either red or green. The anchoring probabilities for (Y,R,G) are $(\frac{1}{5}, \frac{2}{5}, \frac{2}{5})$, but this situation can easily be described as an urn containing balls of five colors.

the risk premium per gamble, $\mu - \frac{c^n}{n}$, do not necessarily go down to zero as n increases to infinity. Since the ambiguity premium is positive, it is obvious that the total premium paid to trade an ambiguous lottery for its expected value too does not have to go to zero as n increases. But we are interested in a different question: What happens to the ambiguity premium per urn the decision maker is willing to pay to avoid the ambiguous act L^n, beyond what he is willing to pay to avoid the non-ambiguous, probabilistic lottery X^n? The aim of this paper is to investigate the connection between $\lim\limits_{n\to\infty} \frac{d^n}{n}$ and $\lim\limits_{n\to\infty} \frac{c^n}{n}$.

We analyze this connection for a given prospect L and its corresponding lottery X.

3   Choquet Expected Utility (CEU)

In this section we consider preferences over ambiguous prospecets that can be represented by the CEU model (Schmeidler [26]). According to this theory, there are utility functions $u^n : \Re \to \Re$ and a capacities $\nu^n : \Omega^n \to [0,1]$ such that $\nu^n(\varnothing) = 0$, $\nu^n(\mathcal{S}^n) = 1$, and the value of L^n, $\mathrm{CEU}^n(L^n)$, is

$$u^n(x^n_{k_n})\nu^n(E^n_{k_n}) + \sum_{i=1}^{k_n-1} u^n(x^n_i)\left[\nu^n\left(\bigcup_{j=i}^{k_n} E^n_j\right) - \nu^n\left(\bigcup_{j=i+1}^{k_n} E^n_j\right)\right] \qquad (1)$$

We assume that all the utility functions $u^1, \ldots, u^n$ are the same and denote them u. Also, we assume that the decision maker is risk averse (hence his vNM utility u is concave) and ambiguity averse in the sense that he prefers playing X^n to playing L^n. To ensure ambiguity aversion we assume that $\nu^n(E) \leqslant P^n(E)$ for all $E \in \Omega^n$, which is equivalent to $P^n \in \mathrm{Core}(\nu^n)$. Note however that we do not require the capacities ν^n to be convex.[3] For exact definitions and analysis of these concepts, see Ghirardato and Marinacci [11] and Chateauneuf and Tallon [1]. See also Machina and Siniscalchi [18].

Ambiguity aversion realizes that the union of two ambigous events can be non-ambiguous. For example, in the 3-color Ellsberg urn, the union of the two ambigous colors leads to an event with probability $\frac{2}{3}$. The contribution of an event to the value of a gamble can therefore be larger than its anchor probability. If there is only a finite number of events, then there is of course

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[3]Convexity of the capacity ν means that $\nu(E) + \nu(E') \leqslant \nu(E \cup E') + \nu(E \cap E')$.

an upper bound to the ratio between the contribution of the capacities generated by all events and their probabilities. Our main requirement is that the following boundedness condition holds uniformally for all n, that is, that the potential over-estimation of the contribution of all events will not go to infinity. Formally:

Boundedness There is K such that for all n and for all disjoint events $E, E' \in \Omega^n$, $\nu^n(E \cup E') - \nu^n(E) \leqslant K P^n(E')$.

This definition is satisfied in a trivial way if the capacity is a probability function and the decision maker is an expected utility maximizer. The following example show that

Example 1 Assume urns with 100 balls each of two colors, G and R. When there are n urns, there are 2^n possible outcomes of the samples (that is, $\{G, R\}^n$), with typical elements $s = (s_1, \ldots, s_{2^n})$, where for all i, $s_i \in \{G, R\}$. The anchor probability P^n of each event E is $|E|2^{-n}$. Define capacities ν^n by

$$\nu^n(E) = \begin{cases} 0 & |E| \leqslant 2^{n-1} \\ \frac{|E| - 2^{n-1}}{2^{n-1}} & |E| > 2^{n-1} \end{cases}$$

Let $K = 2$. By definition, $\nu^n(E \cup E') - \nu^n(E) \leqslant \frac{|E'|}{2^{n-1}} = K P^n(E')$. ∎

Following the discussion of the last section, consider the given non-degenerate random variable L with the anchoring lottery X, and suppose that the decision maker is using the CEU model for ambiguous random variables. Our analysis yields different results when the expected value of X is zero and when it is positive. Consider first the case $\mathrm{E}[X] = 0$. A risk averse decision maker will reject it. And if he dislikes ambiguity and ambiguity is added to the risk, then such a decision maker will certainly reject an ambigous random variable L. The next theorem shows that the average risk premium and the average ambiguity premium converge to the same limit (which may be strictly negative or zero). Its proof, as well as all other proofs, are in the appendix.

Theorem 1 Consider an ambiguous act L with the anchor lottery X such that $\mathrm{E}[X] = 0$. Suppose that the CEU decision maker is risk averse, ambiguity averse, and satisfies boundedness. Then $\lim\limits_{n \to \infty} \frac{d^n}{n} = \lim\limits_{n \to \infty} \frac{c^n}{n}$.

6

Stricter results can be obtained if further assumptions are made regarding the boundedness of the utility function u. As was shown by Fishburn [9, Section 14.1], Savage's [25] axioms imply that the utility function has to be bounded, both from above and from below.

Proposition 1 Suppose that the CEU decision maker is ambiguity averse and satisfies boundedness. If u is bounded from above and from below, then $\lim_{n\to\infty} \frac{c^n}{n} = \lim_{n\to\infty} \frac{d^n}{n} = 0$.

Consider now a different case, where $\mathrm{E}[X] > 0$. This of course doesn't mean that the decision maker would like to play X, or even that if he would like to play it once he would like to play it n times. And it may certainly happen that he would like to play X, but will decline the corresponding random variable L. For example, the decision maker may accept the lottery $(-100, \frac{1}{2}; 110, \frac{1}{2})$, yet decline the gamble where in the two-color Ellsberg urn he wins 110 if he correctly guesses the color of the drawn ball, but loses 100 if he does not. Nevertheless, if for all n, $c^n \geqslant 0$, then the limit of $\frac{d^n}{n}$ is non-negative. Note that this theorem does not require ambiguity aversion.

Theorem 2 Consider an ambiguous act L with the anchor lottery X such that $\mathrm{E}[X] > 0$. Suppose that the CEU decision maker is risk averse and satisfies boundedness. If there exists n_0 such that for all $n \geqslant n_0$, $c^n \geqslant 0$, then $\lim_{n\to\infty} \frac{d^n}{n} \geqslant 0$.

Here too, stricter results can be obtained with further restrictions on the utility function u. Assume first that u is bounded from above, which is used to avoid phenomena in the spirit of the famous St. Petersburg paradox. Proposition 2 shows that if $\lim_{n\to\infty} \frac{c^n}{n} > 0$, then not only is the average certainty equivalent of L^n asymptotically non-negative, but from a certain point on the ambiguos acts L^n become strictly desirable.

Proposition 2 Under the assumptions of Theorem 2, if u is bounded from above and $\lim_{n\to\infty} \frac{c^n}{n} > 0$, then there exists n^* such that $\forall n > n^*$, $L^n \succ 0$.

The next proposition strengthens Theorems 1 and 2 to general lotteries X where u is exponential, thus representing constant risk aversion.

Proposition 3 Suppose that the CEU decision maker is risk averse, ambiguity averse, and satisfies boundedness. If u is exponential, then $\lim\limits_{n\to\infty} \frac{c^n}{n} = \lim\limits_{n\to\infty} \frac{d^n}{n}$.[4]

How restrictive is the boundedness assumption? Formally, are there any bounded capacities that are not expected utility (and therefore, in particular, are different from the anchor probabilities)? And if there are such capacities, does boundedness imply that ν^n converges to a capacity ν with a degenerate core, which is equal to the anchor probability measure? If this is the case, then the boundedness assumption makes the analysis trivial, because the limit of the capacities ν^n is just the anchor probability vector. Example 1 shows however that this is not the case. There are non expected utility bounded capacities for which the core does not converge to a singleton.

Example 1 (cntd) For $s \in \mathcal{S}^n$, define

$$
\tilde{P}^n(s) = \begin{cases}
0 & |\{i : s_i = G\}| < \frac{n}{2} \\
0 & |\{i : s_i = G\}| = \frac{n}{2} \text{ and } s_1 = G \\
\frac{1}{2^{n-1}} & \text{otherwise}
\end{cases}
$$

For each $E \in \Omega^n$, define $\tilde{P}^n(E) = \sum_{s \in E} \tilde{P}^n(s)$. For every E,

$$
\tilde{P}^n(E) \geqslant 2\left(\frac{|s : s \in E|}{2^n} - \frac{1}{2}\right) = \nu^n(E)
$$

Hence $\tilde{P}^n$ is in the core of ν^n and clearly $\tilde{P}^n$ and P^n do not converge to the same limit. ∎

Theorems 1 and 2 do not always hold without the boundedness assumption. For Theorem 1, Let $\nu^n(E) = 1 - \sqrt{1 - P^n(E)}$. This sequence does not satisfy the boundedness assumption. To see why, let $E^{n'} = \{(G, \dots, G)\}$ and let $E^n = \neg E^{n'}$. We obtain

$$
\nu^n(E^n \cup E^{n'}) - \nu^n(E^n) = 1 - \left(1 - \sqrt{1 - \frac{2^n - 1}{2^n}}\right) = \frac{1}{\sqrt{2^n}}
$$

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[4]A sufficient condition for boundedness from above is that the Arrow-Pratt measure of absolute risk aversion is bounded away from 0. That is, that there exists $\delta > 0$ such that for all z, $r_u(z) = -u''(z)/u'(z) > \delta$. To see it, let $v(z) = -e^{-\delta z}$. Then $r_u(z) > r_v(z)$ and, by Pratt [22], there exists a concave h such that $u = h \circ v$. The boundedness of u follows from that of v.

The ratio between this difference and 2^{-n}, the probability of $E^{n\prime}$, is $\sqrt{2^n}$, which is not bounded by any K.

Consider a CEU decision maker with the utility function $u(x) = 1 - e^{-x}$ and the above capacities who is facing the ambiguous act $L = (-0.5, E_1; 0.5, E_2)$ with the anchor lottery $X = (-0.5, \frac{1}{2}; 0.5, \frac{1}{2})$. Calculating c^n and d^n yields $\lim_{n\to\infty} \frac{c^n}{n} = -0.1201$ while, $\lim_{n\to\infty} \frac{d^n}{n} < -0.21$. The same functions applied to the act $L = (-.35, E_1; 0.65, E_2)$ with the anchor lottery $X = (-0.35, \frac{1}{2}; 0.65, \frac{1}{2})$ yield $\lim_{n\to\infty} \frac{d^n}{n} = -0.068 < 0$ while for all n, $\frac{c^n}{n} = 0.030 > 0$ show that Theorem 2 does not hold for all non-Lipschitz functions.

Boundedness is sufficient for theorem 1, but not necessary. Let $\nu^n(E) = P^n(E) - (1 - P^n(E)) \ln(1 - P^n(E))$ which can be shown to be unbounded using the same events E^n and $E^{n\prime}$ as before. Yet numerical analysis shows that when $u(x) = x$, for the ambiguous act L with the anchor lottery $X = (-1, \frac{1}{2}; 1, \frac{1}{2})$, $\lim_{n\to\infty} \left[\frac{c^n}{n} - \frac{d^n}{n} \right] = 0$.

The boundedness of u is required for Proposition 2 as without it it is possible to have $\lim_{n\to\infty} \frac{c^n}{n} > 0$ while for every n^* there is $n \geqslant n^*$ such that $d^n = 0$. See Example 3 in the Appendix.

4   Maxmin EU

Gilboa and Schmeidler [12] suggested the folowing maxmin expected utility (MEU) theory. Under ambiguity, the decision maker behaves as if he has a (convex) set of possible probability distributions as well as a utility function u. For each gamble he computes the values of the expected utility of u with respect to the different possible probability distributions, and evaluates the gamble as the minimum of all these values.

Consider the following example, where each of n urns contains the same number of balls and each of them contains two colours, red and green. The ambiguous act L is $(-1, G; 1, R)$, with the corresponding anchor lottery $X = (-1, \frac{1}{2}; 1, \frac{1}{2})$. Let $s < \frac{1}{2}$, and assume that there are two possible priors for the proportion of green and red balls: $(1 - s, s)$ (the bad urn) and $(s, 1 - s)$ (the good urn).[5] The compositions of the n urns are statistically independent. Following the notation of Section 2, for $i = 1, \ldots, n + 1$, the decision maker

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[5]The results of this section hold even when the basic set of priors $\mathcal{Q}$ is the entire interval $[(1 - s, s), (s, 1 - s)]$, and not just its end points.

wins $2i - n - 2$ if E_i^n happens, which is the event "$i - 1$ red balls and $n - i + 1$ green balls were drawn from the n urns."

A profile of the n urns is an ordered list of the 'good' and 'bad' urns. There are 2^n such profiles, and each of them will induce a probability distribution over $\mathcal{E}^n = \{E_1^n, \ldots, E_{n+1}^n\}$. Observe that two profiles with the same number of 'good' urns induce the same probability distribution. In other words, the decision maker has $n+1$ possible priors over $\mathcal{E}^n$, one for each possible number of 'good' urns, which is an integer between zero and n. Denote this set of priors $\mathcal{Q}^n$.

Unlike the conclusions of Theorems 1 and 2 with respect to the CEU model, in the MEU model the average $\frac{d^n}{n}$ does not necessarily converge to the average $\frac{c^n}{n}$ when $\mathrm{E}[X] = 0$, nor does it become non-negative when $\frac{c^n}{n} \geqslant 0$. First, consider Theorem 1. If the decision maker is risk neutral (that is, $u(x) = x$), then $c^n \equiv 0 \equiv \frac{c^n}{n}$. But as the worst possible prior results from a profile in which all n urns are 'bad' (that is, in each one of them the proportion of red balls is $s < \frac{1}{2}$), the MEU value of playing the n urns is $n[s - (1 - s)] = n[2s - 1]$, and since u is linear this is also the value of d^n. It follows that $\lim_{n \to \infty} \left[\frac{c^n}{n} - \frac{d^n}{n} \right] = 1 - 2s > 0$. To see that Theorem 2 does not hold, consider the ambiguous act $L = (-1 + a, G; 1 + a, R)$ with the corresponding risky lottery $X = (-1 + a, \frac{1}{2}; 1 + a, \frac{1}{2})$, and assume that $0 < a < 1 - 2s$. Then $\frac{c^n}{n} = a > 0$ while $\frac{d^n}{n} = 2s - 1 + a < 0$.

There is a connection between the MEU and CEU models. A capacity ν is convex if for all E, E', $\nu(E) + \nu(E') \leqslant \nu(E \cup E') + \nu(E \cap E')$. For a convex capacity ν, the core $\mathcal{C}_\nu$ of ν is the set of all distributions such that for every E, $Q(E) \geqslant \nu(E)$. The CEU preferences with the capacity ν are the same as the MEU preferences when the set of possible priors is the core of ν (see Schmeidler [26]). Moreover, for every E, $\nu(E) = \min_{Q \in \mathcal{C}_\nu}\{Q(E)\}$.

Define a capacity ν^n by $\nu^n(E_i^n) = \min_{Q \in \mathcal{Q}^n}\{Q(E_i^n)\}$. Since MEU and CEU are equivalent when the capacity is convex, it follows by the above inequality that the boundedness assumption cannot be satisfied. To see why, let $E^{n'} = E_0^n$, the event "all drawn balls are green." Let $E^n = \neg E^{n'}$, the event "having at least one red ball." The probability of this event is minimized when all urns are 'bad' (that is, in all of them there are more green balls than red). In this case, the probability of E_0^n is $(1 - s)^n$, and the probability of $\neg E_0^n$ is $1 - (1 - s)^n$, which is $\nu^n(\neg E_0^n)$. Recall that the anchor probability

10

of G is $\frac{1}{2}$, hence

$$\frac{\nu^n(E^n \cup E^{n\prime}) - \nu^n(E^n)}{P^n(E^{n\prime})} = \frac{1 - \nu^n(\neg E_0^n)}{0.5^n} = \frac{(1-s)^n}{0.5^n}$$

Since $s < \frac{1}{2}$, this ratio is unbounded in n.

Remark: Checking for convexity of ν^n is not trivial. But consider an extreme case — each urn contains one ball, either red or green. In this case $s = 0$ and if the decision maker is willing to consider all possible scenarios then the derived capacity from the multiple priors is given by $\nu^n(E) = 0$ for all $E \neq \mathcal{S}^n$ and $\nu^n(\mathcal{S}^n) = 1$ ($\mathcal{S}^n$ is the sure event). These capacities are trivially convex.

The MEU decision maker is uncertain about the composition of a single urn and therefore, being pessimistic, considers only unfavorable compositions. If he believes that all urns are independent, then the most unfavorable coposition is that in all of them there are more green balls than red. It is therefore not surprising that his ambiguity premium will not disappear.

The beliefs that all urns are 'bad' may be understandable when the decision maker is facing two or three urns. But when he is facing many urns, is it reasonable for him to believe that in *all* of them there are more green balls than red? Such beliefs require an extreme degree of pessimism, and seem less reasonable when more urns are involved.

It is more reasonable to assume that at the presence of many urns, the cautious decision maker may fear that he is unlucky, but not to the extreme level of facing a sequence of urns that are all 'bad.' Suppose for example that he believes that at least $t < \frac{1}{2}$ proportion of the urns are 'good.' This will require a different definition of the basic events, but it turns out that this will not solve the problem. We show that even if these beliefs lead to CEU preferences, the boundedness assumption must be violated, and $\lim_{n \to \infty} [\frac{c^n}{n} - \frac{d^n}{n}]$ may be strictly negative.

Consider the events E_0^n and $\neg E_0^n$. Since beliefs are now on the proportion of 'good' and 'bad' urns, the anchor lottery is obtained from the case where exactly half of the urns are 'good' and half are 'bad.' In this case, the anchor probability of E_0^n, that is, of drawing no red balls, is $[s(1-s)]^{n/2}$. To calculate $\nu(\neg E_0^n)$, note that the worst possible scenario for the decision maker is that only t proportion of the urns are 'good.' In that case the probability that green balls will be drawn from all the tn 'good' urns and from all the $(1-t)n$

11

'bad' urns is $s^{tn}(1-s)^{(1-t)n}$. We obtain

$$\frac{\nu^n(E^n \cup E^{n\prime}) - \nu^n(E^n)}{P^n(E^{n\prime})} = \frac{1 - \nu^n(\neg E_0^n)}{[s(1-s)]^{n/2}} =$$

$$\frac{s^{tn}(1-s)^{(1-t)n}}{[s(1-s)]^{n/2}} = \left(\frac{1-s}{s}\right)^{(\frac{1}{2}-t)n}$$

Since $s < \frac{1}{2}$, this last expression is unbounded.

Continuing with the main example of this section, we show next that $\lim_{n\to\infty} \frac{c^n}{n} = 0$ while $\lim_{n\to\infty} \frac{d^n}{n} < 0$, thus proving that the conclusion of Theorem 1 does not hold. It is indeed easy to verify that when u is linear, $c^n \equiv \frac{c^n}{n} \equiv 0$. The value of d^n is computed with respect to the worst possible scenario, when only t proportion of the urns are 'good.' Using the Binomial distribution we obtain that the expected value of the ambiguous lottery is $d^n = n[t(1-2s) + (1-t)(2s-1)] = n(2t-1)(1-2s)$, hence $\lim_{n\to\infty} \frac{d^n}{n} = (2t-1)(1-2s)$. Since $t < \frac{1}{2}$, this limit is negative.

5   The Smooth Model

Klibanoff, Marinacci, and Mukerji [15] suggested the following smooth case of the recursive model [27]. According to their model, the decision maker has a set of possible probability distributions, and he attaches a probability to each of them. He computes the certainty equivalent of the uncertain act using expected utility with the vNM function u for each of the possible distributions, and then evaluates the lottery over these values using the vNM function ϕ. Ambiguity aversion in this model is reflected by ϕ being more concave than u. Ambiguity neutrality is obtained when ϕ and u are the same.

Formally, let $L = (x_1, E_1; \ldots; x_m, E_m)$ be an ambiguous act, and denote $p = (p_1, \ldots, p_m)$. The decision maker believes that with probability μ^i, $i = 1, \ldots, \ell$, the probability distribution of L is given by $p^i = (p_1^i, \ldots, p_m^i)$. Denote $X_{p^i} = (x_1, p_1^i; \ldots; x_m, p_m^i)$ and let $p = \sum_{i=1}^\ell \mu^i p^i$. Hence, $X = (x_1, p_1; \ldots; x_m, p_m) = \sum_{i=1}^\ell \mu^i X_{p^i}$ is the anchor lottery of L. The value of

L under the smooth model is given by[6]

$$\text{SM}^{\phi u}(L) = \sum_{i=1}^{\ell} \mu^i \cdot \phi \circ u^{-1}\left(\text{EU}^u(X_{p^i})\right)$$

When there is no ambiguity (that is, the decision maker believes that with probability 1 the probability distribution of X is p), then the value of X is $\phi \circ u^{-1}(\text{EU}^u(X))$ which represents the same order as EU with the vNM utility u. Note that $\text{EU}^u(X)$ is the value attached to L by an ambiguity neutral desicion maker for whom $\phi = u$. To see why, observe that

$$\text{SM}^{uu}(L) = \sum_{i=1}^{\ell} \mu^i \cdot \text{EU}^u(X_{p^i}) = \text{EU}^u\left(\sum_{i=1}^{\ell}\mu^i X_{p^i}\right) = \text{EU}^u(X)$$

The certainty equivalents are defined by $u(c^1) = \text{E}[u(X)]$ and $\phi(d^1) = \text{SM}^{\phi u}(L)$.[7] We have

$$u(c^1) = \text{EU}^u(X) = \sum_{i=1}^{\ell} \mu^i \cdot \text{EU}^u\left(X_{p^i}\right)$$

On the other hand,

$$\phi \circ u^{-1}(u(d^1)) = \phi(d^1) = \sum_{i=1}^{\ell} \mu^i \cdot \phi \circ u^{-1}\left(\text{EU}^u\left(X_{p^i}\right)\right)$$

Assuming ambiguity aversion implies that $\phi \circ u^{-1}$ is concave, hence it follows by the definition of risk aversion for expected utility theory that $u(d^1) \leqslant u(c^1)$, hence $d^1 \leqslant c^1$.

As before, let X^n and L^n be n-repetitions of X and L. The value of X^n and its certainty equivalent c^n are given by $u(c^n) = \text{EU}^u(X^n)$.

Consider L^n, which is a sequence of n repetitions of L. A typical possible beliefs of the nature of this sequence is a list of n lotteries, each taken from the set $\{X_{p^1}, \ldots, X_{p^\ell}\}$, where X_{p^i} appears j_i times, $i = 1, \ldots, \ell$, and $\sum_i j_i = n$. The probability of such a sequence is the product of the corresponding μ^i

---

[6]Since this model is using two different vNM functions, we add a superscript index (u or ϕ) to indicate the utility function used in the EU operator.

[7]The certainty equivalent of the smooth model is computed using ϕ since $\text{SM}^{\phi u}(x, s_1; \ldots; x, s_n) = \phi(x)$.

probabilities, that is, $\prod_i (\mu^i)^{j_i}$. There are $(\ell)^n$ (ℓ to the power of n) such possible sequences, denote them $\{Y^n_j\}_{j=1}^{(\ell)^n}$ and denote their corresponding probabilities μ^n_j. We thus obtain that

$$\mathrm{SM}^{\phi u}(L^n) \;=\; \sum_{j=1}^{(\ell)^n} \mu^n_j \cdot \phi \circ u^{-1}(\mathrm{EU}^u(Y^n_j)) \tag{2}$$

and $\phi(d^n) = \mathrm{SM}^{\phi u}(L^n)$.

The next theorem presents conditions under which the results of Theorem 1 hold for the smooth model. Under risk and ambiguity aversion, if the concavity of u and ϕ converge to the same limit as $x \to -\infty$, then the limits of the averages of the risk and the ambiguity certainty equivalents are the same.

Theorem 3 Consider an ambigous act L with the anchor lottery X such that $\mathrm{E}[X] = 0$. Suppose that the SM decision maker is both risk and ambiguity averse and that $\lim\limits_{x \to -\infty} \frac{\phi''(x)}{\phi'(x)} = \lim\limits_{x \to -\infty} \frac{u''(x)}{u'(x)}$. Then $\lim\limits_{n \to \infty} \frac{d^n}{n} = \lim\limits_{n \to \infty} \frac{c^n}{n}$.

Although $\frac{\phi''(x)}{\phi'(x)} \equiv \frac{u''(x)}{u'(x)}$ implies that ϕ is an affine transformation of u, the restriction $\lim\limits_{x \to -\infty} \frac{\phi''(x)}{\phi'(x)} = \lim\limits_{x \to -\infty} \frac{u''(x)}{u'(x)}$ does not imply that in the limit ϕ is an affine transformation of u.

Example 2 Let $u(x) = x$ and

$$\phi(x) = \begin{cases} \frac{1-(x-1)^2}{2} & x \leqslant 0 \\[2mm] \sqrt{2x+1} - 1 & x > 0 \end{cases}$$

The function ϕ is continuous and twice differentiable, where

$$\phi'(x) = \begin{cases} 1 - x & x \leqslant 0 \\[2mm] \frac{1}{\sqrt{2x+1}} & x > 0 \end{cases} \qquad\qquad \phi''(x) = \begin{cases} -1 & x \leqslant 0 \\[2mm] -\frac{1}{(2x+1)^{3/2}} & x > 0 \end{cases}$$

Obviously, $\frac{u''}{u'} \equiv 0$ and $\lim\limits_{x \to -\infty} \frac{\phi''(x)}{\phi'(x)} = 0$, yet ϕ is nowhere an affine transformation of u. In fact, for $x < 0$, ϕ is quadratic while u is everywhere linear.

14

Proposition 2 analyzed conditions under which, under the CEU model, $L^n \succ 0$. The next proposition offers conditions for a similar result under the SM model.

Proposition 4 Consider an ambigous act L with the anchor lottery X such that $\mathrm{E}[X] > 0$. Suppose that the SM decision maker is both risk and ambiguity averse and that $\lim\limits_{x \to -\infty} - \frac{\phi''(x)}{\phi'(x)} = 0$. Then there exists n^* such that $\forall n > n^*$, $L^n \succ 0$.

Theorem 3 assumes that $\lim\limits_{x \to -\infty} - \frac{\phi''(x)}{\phi'(x)} = \lim\limits_{x \to -\infty} - \frac{u''(x)}{u'(x)}$. The next theorem shows that if $\lim\limits_{x \to -\infty} - \frac{\phi''(x)}{\phi'(x)} > 0$, then there is always a sufficiently low upper bound of u under which the limits of the averages of the risk and the ambiguity certainty equivalents are not the same.

We say that the risk aversion of utility function u is bounded from above [from below] by ζ if for all x, $-u''(x)/u'(x)$ is less than [more than] ζ. The next result shows that if the degree of risk aversion of ϕ is bounded from below by $t > 0$, then for u with degree of risk aversion that is bounded from above by a sufficienty small s, the limit of the average premium, $\frac{d^n}{n}$ is strictly less than the limit of the average risk premium, $\frac{c^n}{n}$.

Theorem 4 Consider an ambigous act L with the anchor lottery X such that $\mathrm{E}[X] = 0$. Suppose that the SM decision maker is ambiguity averse and that the risk aversion of ϕ is bounded from below by $t > 0$. Under these assumptions there is $s > 0$ such that if the risk aversion of u is bounded from above by s, then $\lim\limits_{n \to \infty} \frac{d^n}{n} < \lim\limits_{n \to \infty} \frac{c^n}{n}$.

Theorem 5 Consider an ambiguous act L with the anchor lottery X. Suppose that the SM decision maker is strictly ambiguity averse and that both ϕ and u are exponential. Then

$$\lim_{n \to \infty} \frac{d^n}{n} < \lim_{n \to \infty} \frac{c^n}{n}$$

6 Discussion

As early as 1961 did William Fellner [7, pp. 678–9] ask: "there is the question whether, if we observe in him [the decision maker] the trait of nonadditivity,

he is or is not likely gradually to lose this trait *as he gets used to the uncertainty with which he is faced.*" Fellner pointed out a fundamental problem in answering this question empirically: In an experiment, decision makers may understand that the ambiguity is generated by a randomization mechanism and is therefore not ambiguous, but this is not necessarily the case with processes of nature or social life.

Our analysis shows that a lot depends on the way we choose to model ambiguity. But at least within CEU, ambiguity aversion disappears if the decision maker is faced with many similar ambiguous situations. The term "similar" is of course not well defined, but loosely speaking, our analysis shows that even though decision makers don't learn anything new about the world as they face repeated ambiguity, they may still learn not to fear this lack of knowledge. So the doctor of the introduction may learn after seeing many patients to look at the anchoring probability as a guideline for her repeated medical decisions, but it may still be the case that she'll avoid ambiguity and prefer to take risk with known probabilities on her first trip out of the country, even if the anchoring probabilities of the ambiguous option are better than the risky one.

Theorem 1 does not claim that overall ambiguity aversion disappears. It doesn't even rule out the possibility that as the number of incidents n grows, the difference between the certainty equivalents of the anchoring probabilistic lottery and the ambiguous gamble may be unbounded. Similarly, Theorem 2 permits the certainty equivalent of the ambiguous gamble to be unboundedly negative. Both theorems deal with the certainty equivalents per case. An alternative way to analyze attitudes per case is to divide the gamble L^n and the anchoring lottery X^n by n. The probabilistic lottery will then converge to its average. Maccheroni and Marinacci [17] proved that as $n \to \infty$, the capacity of the event "the average outcome of the ambiguous act L is between its CEU (with the linear utility $u(x) = x$) and minus the CEU value of $-L$" is one. Similarly to this extension of the law of large numbers, the central limit theorem of the classical probability was also extended to the uncertainty framework. This was done by Marinacci [19], who used a certain set of capacities, and by Epstein, Kaido, and Seo [5], who made use of belief functions. The latter authors also studies confidence regions.

Very few experiments checked attitudes to repeated ambiguity (although it seems that several more are currently being conducted). Liu and Colman [16] report that participants chose ambiguous options significantly more frequently in repeated-choice than in single-choice. This suggests that repeti-

tion diminishes the effect of ambiguity aversion. Filiz-Ozbay, Gulen, Masatli-oglu, and Ozbay [8] report that ambiguity aversion diminishes with the size of the urn. The intuition behind their result agrees with our finding, since both are based on the idea that the more options there are (number of balls to draw from an urn/number of urns) the less plausible is the extreme pessimistic view that Nature always acts against the decision-maker. Halevy and Feltkamp [13] and Epstein and Halevy [4] conducted experiments that involve drawing from two urns and report that when no information regarding the dependence between the urns is provided, individuals display ambiguity aversion with respect to it. Since we assume that urns are independent, this type of ambiguity is not relevant to the current work.

Other models imply a connection between the CEU and the EU models. Klibanoff [14] studied the relation between stochastic independence and convexity of the capacity in the CEU model and found that together they imply EU (hence the capacity must be additive). His results are not related to ours since we do not assume stochastic independence and, furthermore, the capacities we analyse are not required to be convex.

Appendix: Proofs

Given the anchor lottery $X^n = (x_1^n, p_1^n; \ldots; x_{k_n}^n; p_{k_n}^n)$, define $f^n : [0,1] \to [0,1]$ such that $f^n(0) = 0$, for $i = 1, \ldots, k_n$,

$$f^n \left(\sum_{j=i}^{k_n} p_j^n \right) = \nu^n \left(\bigcup_{j=i}^{k_n} E_j^n \right) \tag{3}$$

and let f^n be piecewise linear on the segment $[0, p_{k_n}^n]$ and on the segments $[\sum_{j=i+1}^{k_n} p_j^n, \sum_{j=i}^{k_n} p_j^n]$, $i = 1, \ldots, k_n - 1$. Note that by ambiguity avresion for all E, $\nu(E) \leqslant \Pr(E)$, hence by the piece-wise linearity of f^n, we have $f^n(p) \leqslant p$. Eq. (1) thus becomes

$$\text{CEU}^n(L^n) = u^n(x_{k_n}^n) f^n(E_{k_n}^n) + \sum_{i=1}^{k_n-1} u^n(x_i^n) \left[f^n \left(\bigcup_{j=i}^{k_n} E_j^n \right) - f^n \left(\bigcup_{j=i+1}^{k_n} E_j^n \right) \right]$$

Denote by F_Z the distribution of lottery Z. In the sequal we use the integral versions of the expected utility and the CEU models. Also, we use

the cummulative (rather than the decummulative) version of the CEU model, defining $g^n(p) = 1 - f^n(1-p)$, to obtain

$$\text{EU}(X^n) = \int u(z)dF_{X^n}(z)$$

$$\text{CEU}(L^n) = \int u(z)dg^n(F_{X^n}(z)) \tag{4}$$

Observe that by the boundedness assumption, for each n, g^n is Lipschitz with K. Also, as for all p, $f^n(p) \leqslant p$, it follows that for all p, $g^n(p) \geqslant p$.

Fact 1 Let $u(x) = -e^{-ax}$. Then for lotteries $X_1, \ldots, X_k$, $\text{EU}(\sum_{i=1}^k X_i) = u(\sum_{i=1}^k \text{CE}(X_i))$, where $\text{CE}(X)$ is the certainty equivalent of X. In particular, for all n, $\frac{c^n}{n} = c^1$.

Proof: Assume wlg that all lotteries have the same possible outcomes, that is, $X_i = (x_1, p_{i1}; \ldots; x_r, p_{ir})$, $i = 1, \ldots, k$. Denote $Y_t = \sum_{i=1}^t X_i$. We show by induction that $\int e^{-az}d(F_{Y_k}(z)) = \prod_{i=1}^k \int e^{-az}dF_{X_i}(z))$. Indeed,

$$\int e^{-az}d(F_{Y_k}(z)) = \sum_{j=1}^r p_{kj} \int e^{-az}d(F_{Y_{k-1}+x_j}(z))$$

$$= \sum_{j=1}^r p_{kj}e^{-ax_j} \int e^{-az}d(F_{Y_{k-1}}(z)) = \int e^{-az}d(F_{Y_{k-1}}(z)) \left(\sum_{j=1}^r p_{kj}e^{-ax_j} \right)$$

$$= \prod_{i=1}^{k-1} \int e^{-az}dF_{X_i}(z)) \times \int e^{-az}dF_{X_k}(z)) = \prod_{i=1}^k \int e^{-az}d(F_{X_i}(z))$$

Therefore

$$\text{EU}(Y_k) = -\int e^{-az}d(F_{Y_k}(z)) = -\prod_{i=1}^k \int e^{-az}dF_{X_i}(z))$$

$$= -\prod_{i=1}^k e^{-a\text{CE}(X_i)} = -e^{-a(\sum_{i=1}^k \text{CE}(X_i))} = u\left(\sum_{i=1}^k \text{CE}(X_i) \right) \qquad \blacksquare$$

Proof of Theorem 1: We prove the theorem through a sequence of claims. Assume throughout, wlg, that $u(0) = 0$ and $u'(0) = 1$.

18

Claim 1 If $\lim_{x \to -\infty} u'(x) = \infty$, then

$$\lim_{n \to \infty} \int_{x>0} u(x) \, dF_{X^n}(x) \bigg/ \int_{x<0} u(x) \, dF_{X^n}(x) = 0$$

Proof: Let $y(\mu) = \sup\{y \leqslant 0 : u(y) < \mu y\}$. Since $\lim_{x \to -\infty} u'(x) = \infty$, it follows that $y(\mu)$ is finite. By the Central Limit Theorem, as $n \to \infty$, the probability that X^n will be in any finite segment goes to 0 yet the probability that it is negative goes to $\frac{1}{2}$. hence $\Pr\{X^n < u^{-1}(\mu x)\} \xrightarrow[n \to \infty]{} \frac{1}{2}$.

Since for positive x, $u'(x) \leqslant 1$, it follows that for such x, $u(x) \leqslant x$. Therefore

$$\frac{\displaystyle\int_{x>0} u(x) \, dF_{X^n}(0)}{\displaystyle\int_{x<0} u(x) \, dF_{X^n}(x)} \geqslant \frac{\displaystyle\int_{x>0} x \, dF_{X^n}(x)}{\displaystyle\int_{x<0} u(x) \, dF_{X^n}(x)}$$

Since $\mathrm{E}(X^n) = 0$, it follows that $\int_{x<0} x \, dF_{X^n}(x) = -\int_{x>0} x \, dF_{X^n}(x)$. Therefore

$$\frac{\displaystyle\int_{x>0} x \, dF_{X^n}(x)}{\displaystyle\int_{x<0} u(x) \, dF_{X^n}(x)} = \frac{\displaystyle-\int_{y(\mu)}^{0} x \, dF_{X^n}(x) - \int_{x<y(\mu)} x \, dF_{X^n}(x)}{\displaystyle\int_{y(\mu)}^{0} u(x) \, dF_{X^n}(x) + \int_{x<y(\mu)} u(x) \, dF_{X^n}(x)} >$$

$$\frac{\displaystyle-\int_{y(\mu)}^{0} x \, dF_{X^n}(x) - \int_{x<y(\mu)} x \, dF_{X^n}(x)}{\displaystyle\int_{y(\mu)}^{0} u(x) \, dF_{X^n}(x) + \mu \times \int_{x<y(\mu)} x \, dF_{X^n}(x)} \xrightarrow[n \to \infty]{} -\frac{1}{\mu}$$

This is true for every $\mu > 1$, hence the claim. $\square$

Claim 2 Suppose that $\lim_{x \to -\infty} u'(x) = \infty$. Then for EU with u and CEU^n with u and f^n, $\lim_{n \to \infty} \frac{\text{CEU}^n(L^n)}{\text{EU}(X^n)} \leqslant K$.

19

Proof: We obtain by claim 1 that

$$\frac{\text{CEU}^n(L^n)}{\text{EU}(X^n)} \leqslant$$

$$\frac{\int_{x<0} u(x)(g^n)'(F_{X^n}(x)) \, dF_{X^n}(x)}{\text{EU}(X^n)} \leqslant$$

$$\frac{K \int_{x<0} u(x) \, dF_{X^n}(x)}{\text{EU}(X^n)} \to K \qquad \qquad \square$$

Claim 3 If $\lim_{x\to-\infty} u'(x)/u(x) < -\ell < 0$, then $\lim_{n\to\infty} \left[\frac{c^n}{n} - \frac{d^n}{n}\right] = 0$.

Proof: Since u is concave, $u(c^n) < 0$. It follows by claim 2 that for $n \geqslant n_0$, $u(d^n) \geqslant (K+1)u(c^n)$. It follows by the concavity of u and by the fact that $d^n \leqslant c^n$ that

$$\frac{u(c^n) - u(d^n)}{c^n - d^n} \geqslant u'(c^n)$$

hence for $n \geqslant n_0$,

$$c^n - d^n \leqslant \frac{u(c^n) - u(d^n)}{u'(c^n)} \leqslant -\frac{Ku(c^n)}{u'(c^n)}$$

Since u is concave, $\lim_{x\to-\infty} u(x) = -\infty$, and as $\lim_{x\to-\infty} u'(x)/u(x) < -\ell < 0$, it follows that $\lim_{x\to-\infty} u'(x) = \infty$. By Fact 2 below, $\lim_{n\to\infty} c^n = -\infty$, hence for a sufficiently large n,

$$-\frac{Ku(c^n)}{u'(c^n)} \leqslant \frac{K}{\ell}$$

Therefore

$$0 \leqslant \frac{c^n}{n} - \frac{d^n}{n} \leqslant \frac{K}{\ell n} \xrightarrow[n\to\infty]{} 0$$

It thus follows that $\lim_{n\to\infty} \left[\frac{c^n}{n} - \frac{d^n}{n}\right] = 0$, which is the claim. $\qquad \square$

Fact 2 If $\lim_{x\to-\infty} u'(x) = \infty$, then $\lim_{n\to\infty} c^n = -\infty$.

Proof: We show that for every integer $m < 0$, $\lim_{n\to\infty} \mathrm{EU}(X^n) \leqslant u(m-1)$. The value of $\mathrm{EU}(X^n)$ equals

$$
\int_{x\leqslant 2(m-1)} u(x)\, dF_{X^n}(x) \left[1 + \frac{\int_{2(m-1)}^{0} u(x)\, dF_{X^n}(x)}{\int_{x\leqslant 2(m-1)} u(x)\, dF_{X^n}(x)} + \frac{\int_{x>0} u(x)\, dF_{X^n}(x)}{\int_{x\leqslant 2(m-1)} u(x)\, dF_{X^n}(x)} \right]
$$

Again by the central limit theorem, $\lim_{n\to\infty} \int_{2(m-1)}^{0} u(x)\, dF_{X^n}(x) = 0$. Also, by the same argument,

$$
\lim_{n\to\infty} \frac{\int_{x>0} u(x)\, dF_{X^n}(x)}{\int_{x\leqslant 2(m-1)} u(x)\, dF_{X^n}(x)} = \lim_{n\to\infty} \frac{\int_{x>0} u(x)\, dF_{X^n}(x)}{\int_{x\leqslant 0} u(x)\, dF_{X^n}(x)}
$$

By Claim 1 the last limit is zero. By the Central Limit Theorem, the probability of receiving an outcome between $2(m-1)$ and 0 converges to zero and the probability of receiving a negative outcome is $\frac{1}{2}$. It thus follows that

$$
\lim_{n\to\infty} \int u(x)\, dF_{X^n}(x) = \lim_{n\to\infty} \int_{x\leqslant 2(m-1)} u(x)\, dF_{X^n}(x) \leqslant \frac{u(2(m-1))}{2} \leqslant u(m-1)
$$

It thus follows that $\lim_{n\to\infty} c^n \leqslant m - 1 < m$. $\qquad\square$

The next two claims deal with the case $\lim_{x\to-\infty} u'(x)/u(x) \to 0$.

Claim 4 If $\lim_{x\to-\infty} u'(x) = H < \infty$, then $\lim_{n\to\infty} \frac{c^n}{n} = \lim_{n\to\infty} \frac{d^n}{n} = 0$.

Proof: Since for all n, $d^n \leqslant c^n$, it is enough to prove that $\lim_{n\to\infty} \frac{d^n}{n} = 0$. Define $v(x) = \min\{Hx, 0\}$. By assumption, $u(x) \geqslant v(x)$ for all x. Let CEU_v^n denote the CEU^n functional with respect to v. Then $\mathrm{CEU}^n(L^n) \geqslant \mathrm{CEU}_v^n(L^n)$ and, as above,

$$
\begin{aligned}
\mathrm{CEU}_v^n(L^n) &= \int v(z)\mathrm{d}g^n(F_{X^n}(z)) \\
&= H \int_{z\leqslant 0} z\mathrm{d}g^n(F_{X^n}(z)) \geqslant KH \int_{z\leqslant 0} z\mathrm{d}F_{X^n}(z)
\end{aligned}
$$

21

Let σ^2 be the variance of X and $n\sigma^2$ the variance of X^n. Note that $E(X^n) = 0$ and choose $\frac{1}{2} < \alpha < 1$. By Chebyshev's inequality,

$$\Pr(X^n < -n^\alpha) \leqslant \frac{n\sigma^2}{n^{2\alpha}} = \frac{\sigma^2}{n^{2\alpha-1}}$$

Assume that n is sufficiently large to satisfy $nx_1 < -n^\alpha$. Then

$$KH \int_{z \leqslant 0} z \mathrm{d}F_{X^n}(z) \;\geqslant\; KH\left(nx_1 \times \frac{\sigma^2}{n^{2\alpha-1}} - n^\alpha \times 1\right)$$

$$= \; KH\left(\frac{x_1\sigma^2}{n^{2(\alpha-1)}} - n^\alpha\right) \;\Longrightarrow$$

$$u(d^n) \;=\; \mathrm{CEU}^n(L^n) \geqslant KH\left(\frac{x_1\sigma^2}{n^{2(\alpha-)}} - n^\alpha\right)$$

and, since u is concave and $u'(0) = 1$,

$$d^n \geqslant KH\left(\frac{x_1\sigma^2}{n^{2(\alpha-1)}} - n^\alpha\right)$$

Therefore,

$$\lim_{n\to\infty} \frac{d^n}{n} \geqslant KH \lim_{n\to\infty}\left(\frac{x_1\sigma^2}{n^{2\alpha-1}} - \frac{1}{n^{1-\alpha}}\right) = 0 \qquad \square$$

Claim 5 If $\lim\limits_{x\to-\infty} u'(x) = \infty$ but $\lim\limits_{x\to-\infty} \frac{u'(x)}{u(x)} = 0$, then $\lim\limits_{n\to\infty} \frac{c^n}{n} = \lim\limits_{n\to\infty} \frac{d^n}{n} = 0$.

Proof: By l'Hospital's rule, $\lim\limits_{x\to-\infty} u''(x)/u'(x) = \lim\limits_{x\to-\infty} u'(x)/u(x) = 0$. Consider the exponential utility $v_\varepsilon(x) = -e^{-\varepsilon x}$ for which $-v_\varepsilon''/v_\varepsilon' \equiv \varepsilon$. Denote by c_ε^n the value of c^n obtained for the function v_ε. By Fact 1, $\lim\limits_{n\to\infty} c_\varepsilon^n/n = c_\varepsilon^1 < 0$ where c_ε^1, the certainty equivalent X, satisfies

$$-e^{-\varepsilon c_\varepsilon^1} = \int -e^{-\varepsilon z}\, dF_X(z) \Longrightarrow c_\varepsilon^1 = -\frac{1}{\varepsilon}\ln\left[\int e^{-\varepsilon z}\, dF_X(z)\right]$$

Note that, using l'Hospital's rule and $E(X) = 0$, $\lim\limits_{\varepsilon\to 0} c_\varepsilon^1 = 0$.

As $\lim_{x \to -\infty} u''(x)/u'(x) = 0$, it follows that for every $\varepsilon > 0$ there is $x(\varepsilon)$ such that for all $x < x(\varepsilon)$, $-u''(x)/u'(x) < \varepsilon$. Define a function u_ε as follows.

$$u_\varepsilon = \begin{cases} u(x) & x \leqslant x(\varepsilon) \\ av_\varepsilon(x) + b & x > x(\varepsilon) \end{cases}$$

where $a = \frac{u'(x(\varepsilon))}{v'_\varepsilon(x(\varepsilon))}$ and $b = u(x(\varepsilon)) - av_\varepsilon(x(\varepsilon))$. Clearly u_ε is less risk averse than v_ε, hence $c^1_{u_\varepsilon} \geqslant c^1_\varepsilon$. By Fact 3 below, $\lim_{n \to \infty} c^n_{u_\varepsilon}/n = \lim_{n \to \infty} c^n/n$. We saw that $\lim_{n \to \infty} c^n_\varepsilon/n = c^1_\varepsilon$, hence $\lim_{n \to \infty} c^n/n \geqslant c^1_\varepsilon$. The claim now follows by the fact that $\lim_{\varepsilon \to 0} c^1_\varepsilon = 0$.

Since u is concave, the fact that (for sufficiently large n) $u(d^n) \geqslant (K + 1)u(c^n)$ implies that $d^n \geqslant (K+1)c^n$. As $\lim_{n \to \infty} \frac{c^n}{n} \to 0$, it follows that $\lim_{n \to \infty} \frac{d^n}{n} \to 0$. $\qquad\square$

Fact 3 If for $x < M$, $u(x) = v(x)$, then $\lim_{n \to \infty} \frac{c^n_u}{n} = \lim_{n \to \infty} \frac{c^n_v}{n}$.

Proof: For $M \geqslant 0$, the fact follows from Claim 1. For $M < 0$, observe that assuming $\lim_{x \to -\infty} u'(x) = \infty$, the probability that X^n is between M and 0 goes to zero with n. $\qquad\square$

Claims 3–5 cover all possible cases of $\lim_{x \to -\infty} u(x)$, hence the theorem. $\qquad\blacksquare$

Proof of Theorem 2: Assume wlg that $n_0 = 1$ and hence $c^n \geqslant 0$ for all n.

First, assume $\lim_{x \to -\infty} u'(x) = \infty$. Define $u^n(x) = u(x) - u(nx_m)$ and note that $u^n(nx_m) = 0$ and $u^n(x) < 0$, for all outcomes of X^n. These inequalities and the boundedness assumption imply that for the CEU^n, the CEU functional with respect to u^n,

$$\begin{aligned} CEU^n(L^n) &= \int u^n(z)dg^n(F_{X^n}(z)) \\ &\geqslant K \int u^n(z)dF_{X^n}(z) \geqslant Ku^n(c^n) \end{aligned}$$

The inequality $u^n(c^n) \geqslant u^n(0)$ yields

$$u^n(d^n) = CEU^n(L^n) \geqslant Ku^n(c^n) \geqslant Ku^n(0)$$

23

Going back to u, noting that $1 - K \leqslant 0$ and that, by concavity, $u(nx_m) \leqslant nu(x_m)$,

$$
\begin{aligned}
u(d^n) &= u^n(d^n) + u(nx_m) \geqslant Ku^n(0) + u(nx_m) \\
&= -Ku(nx_m) + u(nx_m) = (1 - K)u(nx_m) \\
&\geqslant n(1 - K)u(x_m)
\end{aligned}
$$

Denote $A = (1 - K)u(x_m)$. By assumption, $A \leqslant 0$. Note that the concavity of u and $\lim_{x \to -\infty} u'(x) = \infty$ imply $\lim_{y \to -\infty} u^{-1}(y)/y = 0$. Then, $d^n \geqslant u^{-1}(nA)$ implies

$$
\lim_{n \to \infty} \frac{d^n}{n} \geqslant \lim_{n \to \infty} \frac{u^{-1}(nA)}{nA} A = 0
$$

Finally, if $\lim_{x \to -\infty} u'(x) = H < \infty$ ($u'' < 0$ implies that $\lim_{x \to -\infty} u'(x)$ exists), then the proof follows that of Claim 4. ∎

Proof of Proposition 1: If $E[X] = 0$ and u is bounded from above and from below, then $\lim_{n \to \infty} \frac{c^n}{n} = \lim_{n \to \infty} \frac{d^n}{n} = 0$: As in the proof of Theorem 1, we assume that $u(0) = 0$ and $u'(0) = 1$. Let $\hat{u} = \lim_{x \to \infty} u(x)$ and $\check{u} = \lim_{x \to -\infty} u(x)$. First we show that $\lim_{n \to \infty} \frac{c^n}{n} = 0$. Choose $\varepsilon > 0$. Let $\check{x} = u^{-1}(\check{u} + \varepsilon)$ and $\hat{x} = u^{-1}(\hat{u} - \varepsilon)$. Then

$$
EU(X^n) \leqslant \Pr(X^n \leqslant \check{x})(\check{u} + \varepsilon) + \Pr(\check{x} < X^n < \hat{x})(\hat{u} - \varepsilon) + \Pr(X^n \geqslant \hat{x})\hat{u}
$$

and

$$
EU(X^n) \geqslant \Pr(X^n \leqslant \check{x})\check{u} + \Pr(\check{x} < X^n < \hat{x})(\check{u} + \varepsilon) + \Pr(X^n \geqslant \hat{x})(\hat{u} - \varepsilon)
$$

Since $\lim_{n \to \infty} \Pr(\check{x} < X^n < \hat{x}) = 0$ and by the Central Limit Theorem, $\lim_{n \to \infty} \Pr(X^n \leqslant \check{x}) = \lim_{n \to \infty} \Pr(X^n \geqslant \hat{x}) = \frac{1}{2}$ (here we use the fact that $E[X] = 0$),

$$
\frac{\check{u} + \hat{u} - \varepsilon}{2} \leqslant \lim_{n \to \infty} EU(X^n) \leqslant \frac{\check{u} + \hat{u} + \varepsilon}{2}
$$

and since the above holds for all ε, we get

$$
\lim_{n \to \infty} EU(X^n) = \frac{\check{u} + \hat{u}}{2}
$$

24

Therefore, $\lim\limits_{n\to\infty} c^n = u^{-1}\left(\frac{\check{u}+\hat{u}}{2}\right)$ and $\lim\limits_{n\to\infty} \frac{c^n}{n} = 0$.

To prove that $\lim\limits_{n\to\infty} \frac{d^n}{n} = 0$, note that the boundedness of u guarantees the existence of $0 < L < \infty$ satisfying $Lx < u(x)$ for all $x < 0$. Then, define

$$v(x) = \begin{cases} Lx & x \leqslant 0 \\ 0 & x > 0 \end{cases}$$

and proceed as in the proof of Claim 4. ∎

Proof of Proposition 2: If u is bounded from above and $\lim\limits_{n\to\infty} \frac{c^n}{n} > 0$, then there exists n^* such that $\forall n > n_\delta$, $L^n \succ 0$: Without loss of generality, assume that $u(x) < 0$ for all x and that $\lim\limits_{x\to\infty} u(x) = 0$. Similarly to the proof of the first part of Theorem 2,

$$\mathrm{CEU}^n(L^n) = \int u(z)\mathrm{d}g^n(F_{X^n}(z)) \geqslant K \int u(z)\mathrm{d}(F_{X^n}(z)) \geqslant Ku(c^n)$$

Choose δ that satisfies $\lim\limits_{n\to\infty} \frac{c^n}{n} > \delta > 0$ and note that for a sufficiently large n, $c^n > n\delta$. As $n\delta$ goes to infinity, $\lim\limits_{n\to\infty} u(c^n) = 0$ and, by the above argument, $\lim\limits_{n\to\infty} \mathrm{CEU}^n(L^n) = 0$. This implies the existence of n_δ such that for all $n > n_\delta$, $\mathrm{CEU}^n(L^n) > u(0)$. For these n, $L^n \succ 0$. ∎

Proof of Proposition 3: If u is exponential and concave and for all p, $f(p) \leqslant p$, then $\lim\limits_{n\to\infty} \frac{c^n}{n} = \lim\limits_{n\to\infty} \frac{d^n}{n}$: Let $u(x) = -e^{-ax}$, with $a > 0$. By Fact 1, $c^n = nc^1$ and hence $\lim\limits_{n\to\infty} \frac{c^n}{n} = c^1$. By the definitions of c^1 and d^n we have

$$\begin{aligned}
\mathrm{EU}(X - c^1) &= \int -e^{-az}\mathrm{d}F_{X-c^1}(z) = \int -e^{-a(z-c^1)}\mathrm{d}F_X(z) \\
&= e^{ac^1}\int -e^{-az}\mathrm{d}F_X(z) = e^{ac^1}(-e^{-ac^1}) = -1
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
\mathrm{CEU}^n\left(\left(L - \tfrac{d^n}{n}\right)^n\right) &= \int -e^{-az}\mathrm{d}g^n\left(F_{\left(X - \frac{d^n}{n}\right)^n}(z)\right) = \\
\int -e^{-az}\mathrm{d}g^n(F_{X^n - d^n}(z)) &= \int -e^{-a(z-d^n)}\mathrm{d}g^n\left(F_{X^n}(z)\right) = \\
e^{ad^n}\int -e^{-az}\mathrm{d}g^n\left(F_{X^n}(z)\right) &= e^{ad^n}\left(-e^{-ad^n}\right) = -1
\end{aligned} \tag{6}$$

The sequence $\left\{\frac{d^n}{n}\right\}_{n=1}^\infty$ is bounded (since the support of X is) and, by $g(p) \geqslant p$, $\frac{d^n}{n} \leqslant \frac{c^n}{n} = c^1$. Assume, by way of negation, that the sequence does not converge to c^1. Then, there exists $\varepsilon > 0$ and a subsequence $\left\{\frac{d^{n_j}}{n_j}\right\}_{j=1}^\infty$ satisfying $\lim\limits_{j\to\infty} \frac{d^{n_j}}{n_j} < c^1 - \varepsilon$. Without loss of generality, assume that for all j, $\frac{d^{n_j}}{n_j} < c^1 - \varepsilon$. Hence,

$$\mathrm{CEU}\left(\left(L - \tfrac{d^{n_j}}{n_j}\right)^{n_j}\right) = \int -e^{-az}\mathrm{d}g^n\left(F_{(X - d^{n_j}/n_j)^{n_j}}(z)\right)$$

$$> \int -e^{-az}\mathrm{d}g^n\left(F_{(X - c^1 + \varepsilon)^{n_j}}\right)(z) \geqslant -K\int e^{-az}\mathrm{d}F_{(X-c^1+\varepsilon)^{n_j}}(z)$$

$$= -K\left[\int e^{-az}\mathrm{d}F_{X-c^1+\varepsilon}(z)\right]^{n_j} = -Ke^{-an_j\varepsilon}\left[\int e^{-az}\mathrm{d}F_{X-c^1}(z)\right]^{n_j}$$

$$= -Ke^{-an_j\varepsilon} \xrightarrow[j\to\infty]{} 0$$

where the last equality follows by eq. (5). Therefore, for sufficiently large j,

$$\mathrm{CEU}\left(\left(L - \tfrac{d^{n_j}}{n_j}\right)^{n_j}\right) > -1$$

in contradiction with eq. (6). To conclude, $\lim\limits_{n\to\infty} \frac{d^n}{n} = c^1 = \lim\limits_{n\to\infty} \frac{c^n}{n}$. ∎

Example 3 Let $X = (-\frac{1}{4}, \frac{1}{2}; \frac{3}{4}, \frac{1}{2})$. Define $\nu^1 = \dots$ by

$$\nu^n(E) = \begin{cases} 0 & P^n(E) < \frac{1}{2} \\ 2P^n(E) - 1 & \text{otherwise} \end{cases}$$

which is bounded with $K = 2$. We get

$$\mathrm{EU}(X^{4n}) = \sum_{i=-n}^{3n} \binom{4n}{i+n} \frac{1}{2^{4n}} u(i) \tag{7}$$

$$\mathrm{CEU}(L^{4n}) = 2\sum_{i=-n}^{n-1} \binom{4n}{i+n} \frac{1}{2^{4n}} u(i) + \binom{4n}{2n} \frac{1}{2^{4n}} u(n) \tag{8}$$

26

Let $u(x) = x$ for $x \geqslant 0$. We define $u(-n)$ inductively. Let

$$v_n = -\sum_{i=-n+1}^{-1} \binom{4n}{i+n} u(i) - \sum_{i=1}^{n-1} \binom{4n}{i+n} i - \binom{4n}{2n} \frac{n}{2} \qquad (9)$$

$$w_n = 2u(-n+1) - u(-n+2)$$

and define u for $x < 0$ as follows. For $n = 1, \ldots$, let $u(-n) = \min\{v_n, w_n\}$, and for $x \in (-n, -n+1)$ let $u(x) = u(-n) + (x+n)[u(-n+1) - u(-n)]$. The function u is strictly increasing and weakly concave.

Claim 6 $\lim_{n \to \infty} u(-n)/n = -\infty$.

Proof: Suppose not. Then there exists $A > 0$ such that for all n, $-u(-n)/n \leqslant A$, and since between $-n$ and $-n+1$ the function u is linear, it follows that for all n, $-u(-n)/n \leqslant A$.

By definition, $u(-n) \leqslant v_n$, hence it follows by eqs. (8) and (9) that for all n, $\mathrm{CEU}(X^{4n}) \leqslant 0$. On the other hand, by eq. (8),

$$\mathrm{CEU}(X^{4n}) = 2\sum_{i=-n}^{-1} \binom{4n}{i+n} \frac{u(i)}{2^{4n}} + 2\sum_{i=1}^{n-1} \binom{4n}{i+n} \frac{i}{2^{4n}} + \binom{4n}{2n} \frac{n}{2^{4n}}$$

$$\geqslant -\frac{(n-1)nA}{2^{4n-1}} \binom{4n}{n-1} + 1 \times \left[\frac{1}{2} - \Pr(X^{4n} \leqslant 0)\right] \qquad (10)$$

Let $\beta_n = \frac{(n-1)nA}{2^{4n-1}} \binom{4n}{n-1}$. Clearly

$$\frac{\beta_{n+1}}{\beta_n} = \frac{n(n+1)A2^{4n-1} \binom{4n+4}{n}}{(n-1)nA2^{4n+3} \binom{4n}{n-1}}$$

$$= \frac{(n+1)(4n+4)(4n+3)(4n+2)(4n+1)}{16(n-1)n(3n+4)(3n+3)(3n+2)} \to \frac{4^4}{16 \times 3^3} = \frac{16}{27}$$

Hence $\lim_{n \to \infty} \beta_n = 0$. Likewise, $\Pr(X^{4n} \leqslant 0) \leqslant \frac{n}{2^{4n}} \binom{4n}{n} \to 0$, hence the expression of eq. (10) converges to $\frac{1}{2}$; a contradiction. $\qquad \square$

Define $n_0 = 0$, and let n_i satisfy

1. $u(-n_i) = v_{n_i}$

2. For $n_{i-1} < j < n_i$, $u(-j) < v_j$

It follows by Claim 6 that $\{n_i\}$ is not a finite sequence, as otherwise the function u would become linear from a certain point on to the left and will never intersect the line Ax for sufficiently high A.

By definition, $\mathrm{RD}(X^{4n_i}) = 0$. It thus follows by eq. (7) that

$$
\begin{aligned}
c^{4n_i}\left(\tfrac{1}{4}\right) = \mathrm{EU}(X^{4n_i}) \;&=\; \left[\binom{4n_i}{2n_i}\frac{n_i}{2} + \sum_{i=n_i+1}^{3n_i} \binom{4n_i}{i+n_i}i\right]\frac{1}{2^{4n_i}} \\
&>\; \frac{n_i}{2} \times \Pr\left(X^{4n_i} \geqslant n_i\right) = \frac{n_i}{4}
\end{aligned}
$$

Hence $\lim\limits_{i\to\infty} c^{4n_i}/4n_i > \frac{1}{16}$ while $d^{4n_i}/4n_i \equiv 0$. ∎

Proof of Theorem 3 By ambiguity aversion, ϕ is more concave than u, hence $\mathrm{SM}^{\phi\phi}(L^n) \leqslant \mathrm{SM}^{\phi u}(L^n) \leqslant \mathrm{SM}^{uu}(L^n)$. Let $\bar{d}^n$ be the certainty equivalent of L^n under $\mathrm{SM}^{\phi\phi}$ and note that c^n is the certainty equivalent of SM^{uu} (since $\mathrm{SM}^{uu}(L^n) = \mathrm{EU}^u(X^n)$). Hence $\bar{d}^n \leqslant d^n \leqslant c^n$ for all n and

$$
\lim_{n\to\infty} \frac{\bar{d}^n}{n} \leqslant \lim_{n\to\infty} \frac{d^n}{n} \leqslant \lim_{n\to\infty} \frac{c^n}{n}
$$

Using $\mathrm{SM}^{\phi\phi}(L^n) = \mathrm{EU}^\phi(X^n)$, Fact 4 (see below) implies $\lim\limits_{n\to\infty} \frac{\bar{d}^n}{n} = \lim\limits_{n\to\infty} \frac{c^n}{n}$. Hence, $\lim\limits_{n\to\infty} \frac{d^n}{n} = \lim\limits_{n\to\infty} \frac{c^n}{n}$. ∎

Fact 4 Consider a risk averse EU decision maker with a utility u. Let X be a lottery satisfying $\mathrm{E}(X) = 0$ and assume that $\lim\limits_{x\to-\infty} \frac{u''(x)}{u'(x)} = a \in [0,\infty)$. If $a = 0$ then $\lim\limits_{n\to\infty} \frac{c^n}{n} = 0$. Otherwise, $\lim\limits_{n\to\infty} \frac{c^n}{n} = \hat{c}$ where $\hat{c}$ satisfies

$$
-e^{-a\hat{c}} = \int -e^{-az} dF_X(z)
$$

Proof of Fact 4: First assume $a = 0$. When $\lim\limits_{x\to-\infty} u'(x) = H < \infty$, the result follows from Claim 4(proof of Theorem 1). Similarly, when $\lim\limits_{x\to-\infty} u'(x) = \infty$,

28

the result follows from Claim 5 (proof of Theorem 1; note that, by l'Hospital's rule, $\lim_{x\to-\infty} \frac{u''(x)}{u'(x)} = \lim_{x\to-\infty} \frac{u'(x)}{u(x)} = 0$).

Next assume $a > 0$ and note that, in this case, $\lim_{x\to-\infty} u'(x) = \infty$. To see it, note that $\lim_{x\to-\infty} u'(x) = H < \infty$ must imply $\lim_{x\to-\infty} u''(x) = 0$ (by concavity, $u'(x)$ is monotonically icreasing towards H when $x \to -\infty$) and hence $\lim_{x\to-\infty} \frac{u''(x)}{u'(x)} = 0$, contradicting $a > 0$.

Denote $v(x) = -e^{-ax}$ and for a any $\varepsilon > 0$ denote $v_{\varepsilon_+}(x) = -e^{-(a+\varepsilon)x}$, $v_{\varepsilon_-}(x) = -e^{-(a-\varepsilon)x}$ and let $\hat{c}_{\varepsilon_+}$ and $\hat{c}_{\varepsilon_-}$ satisfy

$$-e^{-a\hat{c}_{\varepsilon_+}} = \int -e^{-(a+\varepsilon)z}dF_X(z), \qquad -e^{-a\hat{c}_{\varepsilon_-}} = \int -e^{-(a-\varepsilon)z}dF_X(z)$$

Since v_{ε_+} is more concave than v and v is more concave than v_{ε_-}, we have $\hat{c}_{\varepsilon_+} < \hat{c} < \hat{c}_{\varepsilon_-}$. Let $\hat{c}^n_{\varepsilon_+}$ and $\hat{c}^n_{\varepsilon_-}$ denote the certainty equivalents of X^n under v_{ε_+} and v_{ε_-}, respectively. By Fact 1, $\lim_{n\to\infty} \frac{\hat{c}^n_{\varepsilon_+}}{n} = \hat{c}_{\varepsilon_+}$ and $\lim_{n\to\infty} \frac{\hat{c}^n_{\varepsilon_-}}{n} = \hat{c}_{\varepsilon_-}$.

As $\lim_{x\to-\infty} \frac{u''(x)}{u'(x)} = a > 0$, for every $a > \varepsilon > 0$ there is $x(\varepsilon)$ such that for all $x \leqslant x(\varepsilon)$, $a - \varepsilon < \frac{u''(x)}{u'(x)} < a + \varepsilon$. Define the functions u_{ε_+} and u_{ε_-} by

$$u_{\varepsilon_+}(x) = \begin{cases} u(x) & x \leqslant x(\varepsilon) \\ \\ \alpha_+ v_{\varepsilon_+}(x) + \beta_+ & \text{otherwise} \end{cases}$$

and

$$u_{\varepsilon_-}(x) = \begin{cases} u(x) & x \leqslant x(\varepsilon) \\ \\ \alpha_- v_{\varepsilon_-}(x) + \beta_- & \text{otherwise} \end{cases}$$

where $\alpha_+ = \frac{u'(x(\varepsilon))}{v_{\varepsilon_+}(x(\varepsilon))}$, $\alpha_- = \frac{u'(x(\varepsilon))}{v_{\varepsilon_-}(x(\varepsilon))}$, $\beta_+ = u(x(\varepsilon)) - \alpha_+ v_{\varepsilon_+}(x(\varepsilon))$ and $\beta_- = u(x(\varepsilon)) - \alpha_- v_{\varepsilon_-}(x(\varepsilon))$ are defined as to enable continuity and differentiability of these functions.

Clearly, u_{ε_-} is more risk averse than v_{ε_-} and u_{ε_+} is less risk averse than v_{ε_+}. Hence, $c^n_{u_{\varepsilon_+}}$ and $c^n_{u_{\varepsilon_-}}$, the certainty equivalents of X^n under u_{ε_+} and u_{ε_-}, respectively, satisfy $\hat{c}^n_{\varepsilon_-} \geqslant c^n_{u_{\varepsilon_-}}$ and $c^n_{u_{\varepsilon_+}} \geqslant \hat{c}^n_{\varepsilon_+}$. Hence,

$$\hat{c}_{\varepsilon_-} = \lim_{n\to\infty} \frac{\hat{c}^n_{\varepsilon_-}}{n} \geqslant \lim_{n\to\infty} \frac{c^n_{u_{\varepsilon_-}}}{n} = \lim_{n\to\infty} \frac{c^n}{n} = \lim_{n\to\infty} \frac{c^n_{u_{\varepsilon_+}}}{n} \geqslant \lim_{n\to\infty} \frac{\hat{c}^n_{\varepsilon_+}}{n} = \hat{c}_{\varepsilon_+}$$

29

where the equalities follow from Fact 3.

To conclude, note that both $\hat{c}_{\varepsilon_+}$ and $\hat{c}_{\varepsilon_-}$ converge to $\hat{c}$ when $\varepsilon \to 0$. ∎

Proof of Proposition 4: As in the proof of Theorem 3, let $\bar{d}^n$ be the certainty equivalent of L^n under $\mathrm{SM}^{\phi\phi}$, which is the same as the certainty equivalent of X^n under EU^ϕ. By Proposition 1 in Nielsen [21], for a sufficiently large n, $\bar{d}^n > 0$. The claim now follows from the fact that $\mathrm{SM}^{\phi\phi}(L^n) \leqslant \mathrm{SM}^{\phi u}(L^n)$. ∎

Proof of Theorem 4: If the risk aversion of ϕ is bounded from below by t and u is concave, then for every n, $d^n \leqslant \bar{d}^n$, where $\bar{d}^n$ is the certainty equivalent of L^n obtained from the functions $\bar{u}(x) = x$ and $\phi^*(x) = -e^{-tx}$.

Denote $z_i = \mathrm{E}(X_{p^i})$, $Z = (z_1, \mu^1; ...; z_\ell, \mu^\ell)$ and note that

$$\mathrm{E}(Z) = \sum_{i=1}^{\ell} \mu^i \mathrm{E}(X_{p^i}) = \mathrm{E}\left(\sum_{i=1}^{\ell} \mu^i X_{p^i}\right) = \mathrm{E}(X) = 0$$

If the decision maker is using ϕ^* and $\bar{u}$, then

$$\mathrm{SM}^{\phi^* \bar{u}}(L) = \sum_{i=1}^{\ell} \mu^i \cdot \phi^* \circ \bar{u}^{-1}(\mathrm{EU}^{\bar{u}}(X_{p^i})) = \sum_{i=1}^{\ell} \mu^i \phi^*(\mathrm{E}(X_{p^i}))$$

$$= \sum_{i=1}^{\ell} \mu^i \phi^*(z_i) = \mathrm{EU}^{\phi^*}(Z)$$

Also, it follows from eq. (2) that

$$\mathrm{SM}^{\phi^* \bar{u}}(L^n) = \sum_{j=1}^{\ell^n} \mu_j^n \cdot \phi^* \circ \bar{u}^{-1}(\mathrm{EU}^{\bar{u}}(Y_j^n)) = \sum_{j=1}^{\ell^n} \mu_j^n \phi^*[\mathrm{E}(Y_j^n)]$$

The expected value of Y_j^n is the sum of the expected values of the sequence of lotteries it represents. As there are in this sequence j_i lotteries of type X_{p^i}, $i = 1, \ldots, \ell$, the expected value of Y_j^n is $\sum_{i=1}^{\ell} j_i \mathrm{E}(X_{p^i})$. Hence

$$\sum_{j=1}^{\ell^n} \mu_j^n \phi^*[\mathrm{E}(Y_j^n)] = \sum_{j=1}^{\ell^n} \mu_j^n \phi^* \left[\left(\sum_{i=1}^{\ell} j_i \mathrm{E}(X_{p^i})\right)\right]$$

$$= \sum_{j=1}^{\ell^n} \mu_j^n \phi^* \left[\left(\sum_{i=1}^{\ell} j_i z_i\right)\right] = \mathrm{EU}^{\phi^*}(Z^n)$$

30

Where the last equation follows by the fact that $\sum_{i=1}^{\ell} j_i z_i$ is an outcome of the lottery Z^n which is obtained from playing n times lottery Z. We obtain that

$$\bar{d}^n = (\phi^*)^{-1}(\mathrm{SM}^{\phi^*\bar{u}}(L^n)) = (\phi^*)^{-1}(\mathrm{EU}^{\phi^*}(Z^n))$$

And since ϕ^* is exponential, Fact 1 implies $\frac{\bar{d}^n}{n} = \bar{d}^1 = (\phi^*)^{-1}(\mathrm{EU}^{\phi^*}(Z)) < 0$.

Consider the utility function $u^*(x) = -e^{-sx}$. Since this function represents constant absolute risk aversion, it follows that for this function, the average certainty equivalent of X^n, $\frac{\bar{c}^n}{n}$, equals the certainty equivalent of X, $\bar{c}^1$, which is given by

$$-e^{-s\bar{c}^1} = -\sum p_i e^{-sx_i} \implies \bar{c}^1 = -\ln\left(\sum \frac{p_i}{e^{sx_i}}\right) \Big/ s$$

As $s \to 0$, p_i/e^{sx_i} gets close to p_i, and as $\ln 1 = 0$, the numerator converges to zero. To compute the value of $\bar{c}^1$ as $s \to 0$, use l'Hopital's rule together with the fact that $\sum p_i x_i = 0$ to get

$$\lim_{s \to 0} -\ln\left(\sum \frac{p_i}{e^{sx_i}}\right) \Big/ s = \lim_{s \to 0} \frac{\sum p_i x_i e^{-sx_i}}{\sum \frac{p_i}{e^{sx_i}}} = 0$$

If u is less risk averse than u^*, then $\lim_{n \to \infty} \frac{c^n}{n}$ computed with respect to u will be at least as high as that of u^*. By the first part of the proof $\lim_{n \to \infty} \frac{d^n}{n} \leqslant \bar{d}^1 < 0$. The claim of the theorem now follows from the fact that for sufficiently small s we can get $\bar{c}^1$ as close as we wish to zero, and in particular, for small s, $\lim_{n \to \infty} \frac{d^n}{n} \leqslant \lim_{n \to \infty} \frac{\bar{d}^n}{n} = \bar{d}^1 < \bar{c}^1 = \lim_{n \to \infty} \frac{\bar{c}^n}{n} \leqslant \lim_{n \to \infty} \frac{c^n}{n}$. ∎

Proof of Theorem 5: By construction,

$$u\left(c^1\right) = \mathrm{EU}^u(X) = \mathrm{EU}^u\left(\sum_{i=1}^{\ell} \mu^i X_{p^i}\right) = \sum_{i=1}^{\ell} \mu^i \mathrm{EU}^u\left(X_{p^i}\right)$$

and

$$\left(\phi \circ u^{-1}\right)\left(u\left(d^1\right)\right) = \phi\left(d^1\right) = \sum_{i=1}^{\ell} \mu^i \left(\phi \circ u^{-1}\right)\left(\mathrm{EU}^u\left(X_{p^i}\right)\right)$$

Rewriting the equations and denoting $h = \phi \circ u^{-1}$ yields

31

$$u\left(c^1\right) = \sum_{i=1}^{\ell} \mu^i \mathrm{EU}^u\left(X_{p^i}\right)$$

$$h\left(u\left(d^1\right)\right) = \sum_{i=1}^{\ell} \mu^i h\left(\mathrm{EU}^u\left(X_{p^i}\right)\right)$$

Let $\phi(x) = -e^{-tx}$ and $u(x) = -e^{-sx}$. By strict ambiguity aversion ϕ is strictly more concave than u, which implies $t > s$, and hence $h = \phi \circ u^{-1}$ is strictly concave $(h(y) = -(-y)^{t/s})$. Therefore, these equations imply $u\left(d^1\right) < u\left(c^1\right)$ which, noting that h is increasing, yields $d^1 < c^1$.

By Fact 1, $\frac{c^n}{n} = c^1$ for all n and hence $\lim_{n\to\infty} \frac{c^n}{n} = c^1$. Moreover, denoting $c_i = u^{-1}\left(\mathrm{EU}^u\left(X_{p^i}\right)\right)$ and using Fact 1, any sequence of lotteries $Y_j^n = (X_{p^1})^{n^1} \cdots (X_{p^\ell})^{n^\ell}$, $n^i \in \{0, \mathbb{N}\}$, satisfies

$$
\begin{aligned}
\mathrm{EU}^u\left((X_{p^1})^{n^1} \cdots (X_{p^\ell})^{n^\ell}\right) &= -\mid \mathrm{EU}^u\left(X_{p^1}\right)\mid^{n^1} \cdots \mid \mathrm{EU}^u\left(X_{p^\ell}\right)\mid^{n^\ell} \\
&= -\left(e^{-sc_1}\right)^{n^1} \cdots \left(e^{-sc_\ell}\right)^{n^\ell} \\
&= -e^{-s\left(n^1 c_1 + \ldots + n^\ell c_\ell\right)} \\
&= u\left(n^1 c_1 + \ldots + n^\ell c_\ell\right)
\end{aligned}
$$

Next, denoting $C = \left(c_1, \mu^1; \ldots; c_\ell, \mu^\ell\right)$, $\mathrm{SM}^{\phi u}\left(L^n\right)$ can be written as $\mathrm{EU}^\phi\left(C^n\right)$ for all n:

$$\mathrm{SM}^{\phi u}(L) = \sum_{i=1}^{\ell} \mu^i \phi\left[u^{-1}\left(\mathrm{EU}^u\left(X_{p^i}\right)\right)\right] = \sum_{i=1}^{\ell} \mu^i \phi\left(c_i\right) = \mathrm{EU}^\phi(C)$$

$$\mathrm{SM}^{\phi u}(L^n) = \sum_{j=1}^{(\ell)^n} \mu_j^n \phi\left[u^{-1}\left(\mathrm{EU}^u\left(Y_j^n\right)\right)\right] = \sum_{j=1}^{(\ell)^n} \mu_j^n \phi\left[n^1 c_1 + \ldots + n^\ell c_\ell\right] = \mathrm{EU}^\phi(C^n)$$

Using $d^1 = \phi^{-1}\left(\mathrm{EU}^\phi(C)\right)$ and $d^n = \phi^{-1}\left(\mathrm{EU}^\phi(C^n)\right)$, Fact 1 implies $\frac{d^n}{n} = d^1$ for all n and hence $\lim_{n\to\infty} \frac{d^n}{n} = d^1$. This concludes the proof. $\blacksquare$

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